

Spectral analysis for cascade-emission-based quantum communication in atomic ensembles

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Abstract. The ladder configuration of atomic levels provides a source for telecom photons (signal) from the upper atomic transition. For rubidium and cesium atoms, the signal field has the range around $1.3\text{-}1.5\ \mu\text{m}$ that can be coupled to an optical fiber and transmitted to a remote location. Cascade emission may result in pairs of photons, the signal entangled with the subsequently emitted infrared photon (idler) from the lower atomic transition. This correlated two-photon source is potentially useful in the (Duan-Lukin-Cirac-Zoller) DLCZ protocol for the quantum repeater. We implement the cascade emission to construct a modified DLCZ quantum repeater and investigate the role of time-frequency entanglement in the protocol. The dependence of protocol on photon-number resolving and non-resolving detectors is also studied. We find that frequency entanglement deteriorates the performance but the harmful effect can be diminished by using shorter pump pulses to generate the cascade emission. An optimal cascade-emission-based DLCZ scheme is realized by applying a pure two-photon source in addition to using detectors of perfect quantum efficiency.

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1. Introduction

Quantum communication has opened up the possibility to transmit quantum information over long distance. A quantum repeater protocol proposed by Briegel *et al.* [1, 2] fulfills such a long distance system. Subsequently, Duan, Lukin, Cirac, and Zoller (DLCZ) [3] suggested a long distance quantum communication based on atomic ensembles. This scheme involves Raman scattering of an incoming light from the atoms with the emission of a signal photon. The photon is then correlated with coherent excitation of the atomic ensemble. The information may be transferred through light to another atomic ensemble or retrieved by a reverse Raman scattering process, generating an idler photon directional correlated with the signal one [4, 5, 6, 7, 8]. The signal and idler photons in alkali gases are in the near-infrared spectral region, which mismatches the telecommunication bandwidth optical fiber. Therefore, an alternative process that is able to generate telecom wavelength photons correlated with atomic spin excitations [9, 10, 11] would provide the essential step toward practical long distance quantum communication.

The alkali atomic cascade transition shown in figure 1 is able to generate telecom wavelength light, the signal, from the upper transition and a near-infrared field, the idler, from the lower one. The telecom light can travel through the fiber with minimal loss, while the near-infrared field is suitable for storage and retrieval in an atomic quantum memory element. Their use in a quantum information system requires quantum correlations between stored excitations and the telecom field. It is interesting to assess the cascade scheme in the DLCZ protocol given that it could potentially reduce transmission losses in a quantum telecommunication system.

Correlated photon pairs may be generated by parametric down conversion (PDC) [12, 13, 14]. The degree of entanglement can be quantified by Schmidt mode decomposition [15, 16], allowing the influence of group-velocity matching [17] to be assessed. A pure single photon source is a basis element for quantum computation by linear optics (LOQC) [18], and it can be conditionally generated by measurement [19]. A similar approach can be applied to the study of the transverse degrees of freedom in type-II PDC [20] and PDC in a distributed microcavity [21]. In photonic-crystal fiber (PCF), a factorizable photon pair can be generated by spectral engineering [22]. The spectral effect has been discussed in relation to a quantum teleportation protocol [23] as a first step toward quantum communication.

This motivates the research in this article where we study the spectral effect of correlated photon pair generated from cascade atomic ensemble in DLCZ scheme. The DLCZ scheme is based on entanglement generation and swapping, and quantum state transfer, which make up the basic elements for long distance quantum communication. Generating entanglement is the first step in quantum information processing, and entanglement swapping is the essence to distribute the entanglement over distant places. Quantum state transfer enables the secure transmission to eavesdropping which is therefore of great practical interest [24, 25, 26].

In this article, we start from formulating a two-photon state generated in a cold atomic ensemble. We use Schrödinger's equation to investigate the correlated signal and idler photons spontaneously emitted from two driving lasers via four-level atomic structure. The essence of phase-matching in the four-wave mixing (FWM) conditions is discussed in section 2, and we calculate the second-order correlation function to show the bunching behavior of the cascade-emitted photons. In section 3, we briefly review the Schmidt decomposition that is used to analyze the frequency

entanglement and mode functions of the two-photon source. We characterize these spectral properties of the correlated two-photon state for different superradiant decay constants and study how the laser excitation pulse modifies their spectral profile. We then demonstrate the modified DLCZ scheme, a quantum repeater protocol, which employs cascade emission in section 4. We reconstruct the elements of DLCZ scheme including entanglement generation, entanglement swapping, effective ‘polarization’ maximally entangled state projection, and quantum teleportation. We investigate how frequency entanglement of the cascade photon pair influences these elements, and study their performances (fidelity, heralding and success probabilities) for two types of photon detectors (resolving photon number or not) along with dependence of quantum efficiency. We then conclude in section 5 and discuss the alternative method of generating telecom photon by frequency conversion. The details of the Hamiltonian and Schrödinger’s equation are described in appendix A. In appendix B, we derive the multimode two-photon state and conditional output density operators used in modified DLCZ protocol.

2. A correlated two-photon state

We consider N cold atoms that are initially prepared in the ground state interacting with four independent electromagnetic fields. As shown in figure 1, two driving lasers (of Rabi frequencies Ω_a and Ω_b) excite a ladder configuration $|0\rangle \rightarrow |1\rangle \rightarrow |2\rangle$. Two quantum fields, signal \hat{a}_s and idler \hat{a}_i , are generated spontaneously. The four atomic levels can be chosen as $(|0\rangle, |1\rangle, |2\rangle, |3\rangle) = (|5S_{1/2}, F=3\rangle, |5P_{3/2}, F=4\rangle, |4D_{5/2}, F=5\rangle, |5P_{3/2}, F=4\rangle)$ [9]. The atoms adiabatically follow the two excitation pulses and decay through the cascade emission of signal and idler photons. Based on the discussion in appendix A, we permit only single atomic excitations under the condition of large detuning, $\Delta_1 \gg \sqrt{N}\Omega_a/2$. The Hamiltonian and the coupled equations of the atomic dynamics are detailed in appendix A.

To correctly describe the frequency shifts arising from dipole-dipole interactions, we need to include the non-rotating wave approximation (non-RWA) terms in the electric dipole interaction Hamiltonian. In appendix A, we consider only RWA terms for simplicity and the non-RWA terms would allow virtual transitions that in effect add to the frequency shifts in an appropriate way [27, 28]. The frequency shift has contributions from the single atom Lamb shift and a collective frequency shift. The Lamb shift is assumed to be renormalized into the single atom transition frequency distinguishing it from the collective shift due to the atom-atom interaction.

Writing the state-vector $|\psi(t)\rangle$ in a basis restricted to single atomic excitations, and single pairs of signal and idler photons, we can introduce the probability amplitudes,

$$C_{s,k_i}(t) = \sum_{\mu=1}^N e^{-i\vec{k}_i \cdot \vec{r}_\mu} \langle 3_\mu, 1_{k_s, \lambda_s} | \psi(t) \rangle, \quad (1)$$

and

$$D_{s,i}(t) = \langle 0, 1_{k_s, \lambda_s}, 1_{k_i, \lambda_i} | \psi(t) \rangle, \quad (2)$$

where \vec{k}_i here is equivalent to \vec{q}_i defined in appendix A and is denoted as $\vec{k}_a + \vec{k}_b - \vec{k}_s$ from FWM condition which will be demonstrated later. Note that $C_{s,k_i}(t)$ is an

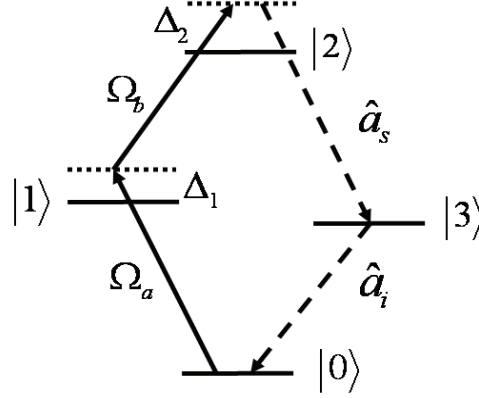


Figure 1. Four-level atomic ensemble interacting with two driving lasers (solid) with Rabi frequencies Ω_a and Ω_b . Cascade emissions, signal and idler fields, are labelled by \hat{a}_s and \hat{a}_i , respectively and Δ_1 and Δ_2 are single and two-photon laser detunings.

amplitude for a phased excitation of the ensemble of atoms subsequent to signal photon emission and $(k_{s,i}, \lambda_{s,i})$ represent wave vectors and polarization indices for signal and idler fields respectively.

After adiabatically eliminating the laser excited levels in the equations of motion, we are able to simplify and derive the amplitude C_{s,k_i} and the signal-idler (two-photon) state amplitude $D_{s,i}$, as shown in appendix A,

$$C_{s,k_i}(t) = g_s^*(\epsilon_{k_s,\lambda_s}^* \cdot \hat{d}_s) \sum_{\mu} e^{i\Delta\vec{k} \cdot \vec{r}_{\mu}} \int_0^t dt' e^{i(\omega_s - \omega_{23} - \Delta_2)t'} e^{(-\frac{\Gamma_3^N}{2} + i\delta\omega_i)(t-t')} b(t'), \quad (3)$$

$$D_{s,i}(t) = g_i^* g_s^*(\epsilon_{k_i,\lambda_i}^* \cdot \hat{d}_i)(\epsilon_{k_s,\lambda_s}^* \cdot \hat{d}_s) \sum_{\mu} e^{i\Delta\vec{k} \cdot \vec{r}_{\mu}} \int_0^t \int_0^{t'} dt'' dt' e^{(-\frac{\Gamma_3^N}{2} + i\delta\omega_i)(t'-t'')} e^{i(\omega_i - \omega_3)t'} e^{i(\omega_s - \omega_{23} - \Delta_2)t''} b(t''), \quad (4)$$

where $b(t) = \frac{\Omega_a(t)\Omega_b(t)}{4\Delta_1\Delta_2}$ is proportional to the product of the Rabi frequencies. Coupling constants $g_{s(i)}$, polarization direction $\epsilon_{k_{s(i)},\lambda_{s(i)}}$, and unit direction of dipole operators $\hat{d}_{s,i}$ are for signal and idler fields respectively. Various definition of optical frequencies ω 's and laser detuning Δ_2 can be found in Appendix A.

The factor $\sum_{\mu} e^{i\Delta\vec{k} \cdot \vec{r}_{\mu}}$ reflects phase-matching of the interaction under conditions of four-wave mixing when the wavevector mismatch $\Delta\vec{k} = \vec{k}_a + \vec{k}_b - \vec{k}_s - \vec{k}_i \rightarrow 0$. The radiative coupling between atoms results in the appearance of the superradiant decay constant

$$\Gamma_3^N = (N\bar{\mu} + 1)\Gamma_3, \quad (5)$$

where Γ_3 is the natural decay rate of the $|3\rangle \rightarrow |0\rangle$ transition, and $\bar{\mu}$ is a geometrical constant depending on the shape of the atomic ensemble. An expression for the collective frequency shift $\delta\omega_i$ is given in the Appendix A. For a cylindrical atomic ensemble, the decay factor $N\bar{\mu} + 1$ depends on the height and radius as shown in equation (A.17). $N\bar{\mu} + 1 \approx 4$ and 6 which are comparable to the operating conditions of the experiment [9].

We use normalized Gaussian pulses as an example where $\Omega_a(t) = \frac{1}{\sqrt{\pi}\tau} \tilde{\Omega}_a e^{-t^2/\tau^2}$, $\Omega_b(t) = \frac{1}{\sqrt{\pi}\tau} \tilde{\Omega}_b e^{-t^2/\tau^2}$, so that the two pulses are overlapped with the same pulse width. $\tilde{\Omega}_{a,b}$ is the pulse area, and let $\Delta\omega_s \equiv \omega_s - \omega_{23} - \Delta_2 - \delta\omega_i$, $\Delta\omega_i \equiv \omega_i - \omega_3 + \delta\omega_i$. In the long time limit, we have the probability amplitude D_{si} ,

$$D_{si}(\Delta\omega_s, \Delta\omega_i) = \frac{\tilde{\Omega}_a \tilde{\Omega}_b g_i^* g_s^* (\epsilon_{k_i, \lambda_i}^* \cdot \hat{d}_i)(\epsilon_s^* \cdot \hat{d}_s)}{4\Delta_1 \Delta_2} \frac{\sum_{\mu} e^{i\Delta\vec{k} \cdot \vec{r}_{\mu}} e^{-(\Delta\omega_s + \Delta\omega_i)^2 \tau^2 / 8}}{\sqrt{2\pi}\tau} \frac{\frac{\Gamma_3^N}{2} - i\Delta\omega_i}{\Gamma_3^N/2 - i\Delta\omega_i}, \quad (6)$$

indicating a spectral width $\Gamma_3^N/2$ for idler photon in a Lorentzian distribution modulating a Gaussian profile with a spectral width $2\sqrt{2}/\tau$ for signal and idler. Energy conservation of signal and idler photons with driving fields at their central frequencies corresponds to $\omega_s + \omega_i = \omega_a + \omega_b$, which makes $\Delta\omega_s + \Delta\omega_i = 0$; the collective frequency shifts cancel.

Using the asymptotic form of the two-photon state given in equation (6), the second-order correlation function [29] is calculated as

$$G_{s,i}^{(2)} = \langle \psi(\infty) | \hat{E}_s^-(\vec{r}_1, t_1) \hat{E}_i^-(\vec{r}_2, t_2) \hat{E}_i^+(\vec{r}_2, t_2) \hat{E}_s^+(\vec{r}_1, t_1) | \psi(\infty) \rangle \quad (7)$$

where

$$\hat{E}_s^+(\vec{r}_1, t_1) = \sum_{k_s, \lambda_s} \sqrt{\frac{\hbar\omega_s}{2\epsilon_0 V}} \hat{a}_{k_s, \lambda_s} \vec{e}_{k_s, \lambda_s} e^{i\vec{k}_s \cdot \vec{r}_1 - i\omega_s t_1}, \quad (8)$$

$$\hat{E}_i^+(\vec{r}_2, t_2) = \sum_{k_i, \lambda_i} \sqrt{\frac{\hbar\omega_i}{2\epsilon_0 V}} \hat{a}_{k_i, \lambda_i} \vec{e}_{k_i, \lambda_i} e^{i\vec{k}_i \cdot \vec{r}_2 - i\omega_i t_2}. \quad (9)$$

$|\psi(\infty)\rangle$ denotes the state vector in the long time limit that involves the ground state and two-photon state vectors. Free electromagnetic fields, signal and idler photons, at space (\vec{r}_1, \vec{r}_2) and time (t_1, t_2) are \hat{E}_s^+ and \hat{E}_i^+ where $(+)$ denotes their positive frequency part. For second order correlation function, only D_{si} contributes to it, then following the standard procedure of $G_{s,i}^{(2)}$ calculation [29], we have

$$\sqrt{G_{s,i}^{(2)}} \propto \sum_{\mu} e^{i\Delta\vec{k} \cdot \vec{r}_{\mu}} e^{-2(\Delta t_s)^2/\tau^2} e^{-\frac{\Gamma_3^N}{2}(\Delta t_i - \Delta t_s)} \Theta(\Delta t_i - \Delta t_s), \quad (10)$$

where $\Delta t_s \equiv t_1 - \frac{\vec{r}_1 \cdot \hat{k}_s}{c}$ and $\Delta t_i \equiv t_2 - \frac{\vec{r}_2 \cdot \hat{k}_i}{c}$. The step function Θ shows the causal connection between signal and idler emissions and is due to the complex integral with the pole at $\Delta\omega_i = -i\frac{\Gamma_3^N}{2} - \delta\omega_i$ in the lower half plane. The emission time for the signal field $(t_1 - \frac{\vec{r}_1 \cdot \hat{k}_s}{c})$ is within the pulse envelope of width τ , and the idler photon decays with a superradiant constant $\Gamma_3^N/2$.

If we let $\Delta t \equiv \Delta t_i - \Delta t_s$ and choose $\Delta t_s = 0$ as the origin in time (idler gating time), then we have the second-order correlation function

$$G_{s,i}^{(2)}(\Delta t) = |\Phi_{s,i}(\Delta t)|^2 \propto e^{-\Gamma_3^N \Delta t} \text{ where } \Delta t \geq 0. \quad (11)$$

It resembles the result for the second-order correlation function in the case of single atom, whereas here we have an enhanced decay rate due to the atomic dipole-dipole interaction. This exponential correlation function indicates the bunching property of cascade photons [30] showing an immediate emission of idler photon following the signal one.

3. Schmidt decomposition

We would like to perform an analysis of entanglement properties of our cascade emission source. In addition to polarization entanglement, a characterization of frequency space entanglement is required to clarify its suitability in the DLCZ protocol [3].

In the long time limit, the state function is given by

$$|\psi\rangle = |0, \text{vac}\rangle + \sum_{s,i} D_{s,i} |0, 1_{\vec{k}_s, \lambda_s}, 1_{\vec{k}_i, \lambda_i}\rangle \quad (12)$$

where $D_{s,i}$ can be found in equation (6) and $|0, \text{vac}\rangle$ is the joint atomic ground and photon vacuum state. Shorthand notations $s = (k_s, \lambda_s)$ and $i = (k_i, \lambda_i)$ are for different spatial modes $k_{s(i)}$ and two degree of freedom polarizations $\lambda_{s(i)}$.

The spatial correlation of two-photon state in FWM condition can be eliminated by pinholes or by coupling to single mode fiber so we consider only the continuous frequency space. For some specific polarizations λ_s and λ_i , we have the state vector $|\Psi\rangle$,

$$|\Psi\rangle = \int f(\omega_s, \omega_i) \hat{a}_{\lambda_s}^\dagger(\omega_s) \hat{a}_{\lambda_i}^\dagger(\omega_i) |0\rangle d\omega_s d\omega_i, \quad (13)$$

where

$$f(\omega_s, \omega_i) = \frac{e^{-(\Delta\omega_s + \Delta\omega_i)^2 \tau^2 / 8}}{\frac{\Gamma_3^N}{2} - i\Delta\omega_i}. \quad (14)$$

Following the theoretical work on two-photon pulses generated from down-conversion by Law *et al.* [15], the quantification of entanglement can be determined in the Schmidt basis where the state vector is expressed as

$$|\Psi\rangle = \sum_n \sqrt{\lambda_n} \hat{b}_n^\dagger \hat{c}_n^\dagger |0\rangle, \quad (15)$$

$$\hat{b}_n^\dagger \equiv \int \psi_n(\omega_s) \hat{a}_{\lambda_s}^\dagger(\omega_s) d\omega_s, \quad (16)$$

$$\hat{c}_n^\dagger \equiv \int \phi_n(\omega_i) \hat{a}_{\lambda_i}^\dagger(\omega_i) d\omega_i, \quad (17)$$

where $\hat{b}_n^\dagger, \hat{c}_n^\dagger$ are effective creation operators and λ_n 's (no confusion with polarization index $\lambda_{s,i}$) are probabilities in corresponding two-photon emission modes n . If $\lambda_1 = 1$, it means a pure two-photon emission. Eigenvalues λ_n , and eigenfunctions ψ_n and ϕ_n , are the solutions of the eigenvalue equations,

$$\int K_1(\omega, \omega') \psi_n(\omega') d\omega' = \lambda_n \psi_n(\omega), \quad (18)$$

$$\int K_2(\omega, \omega') \phi_n(\omega') d\omega' = \lambda_n \phi_n(\omega), \quad (19)$$

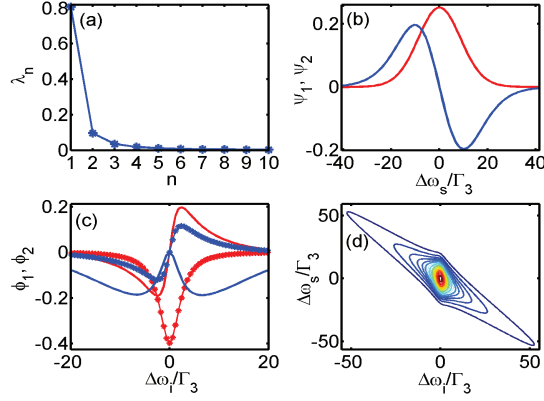


Figure 2. Schmidt mode analysis with pulse width $\tau = 0.25$ and superradiance decay factor $N\bar{\mu} + 1 = 5$. (a) Schmidt number and (b) signal mode functions: $\text{Re}[\psi_1]$ (solid-red) and $\text{Re}[\psi_2]$ (solid-blue). Imaginary parts are not shown, then are zero. (c) Real (solid) and imaginary (dotted) parts of first (red) and second (blue) idler mode functions, ϕ_1 and ϕ_2 . (d) The absolute spectrum $|f(\Delta\omega_s, \Delta\omega_i)|$.

where $K_1(\omega, \omega') \equiv \int f(\omega, \omega_1) f^*(\omega', \omega_1) d\omega_1$ and $K_2(\omega, \omega') \equiv \int f(\omega_2, \omega) f^*(\omega_2, \omega') d\omega_2$ are the kernels for the one-photon spectral correlations [15, 16]. Orthogonality of eigenfunctions is $\int \psi_i(\omega) \psi_j(\omega) d\omega = \delta_{ij}$, $\int \phi_i(\omega) \phi_j(\omega) d\omega = \delta_{ij}$, and the normalization of quantum state requires $\sum_n \lambda_n = 1$.

In the Schmidt basis, the von Neumann entropy may be written

$$S = - \sum_{n=1}^{\infty} \lambda_n \ln \lambda_n. \quad (20)$$

If there is only one non-zero Schmidt number $\lambda_1 = 1$, the entropy is zero, which means no entanglement and a factorizable state. For more than one non-zero Schmidt number, the entropy is larger than zero and bipartite entanglement is present.

The kernel in equation (14) has all the frequency entanglement information, entanglement means $f(\omega_s, \omega_i)$ cannot be factorized in the form $g(\omega_s)h(\omega_i)$, a multiplication of two separate spectral functions. By inspection the Gaussian profile of signal and idler emission is a source of correlation. The joint spectrum $\Delta\omega_s + \Delta\omega_i$ is confined within the width of order of $1/\tau$. The Lorentzian factor associated with the idler emission has a width governed by the superradiant decay rate.

In figure 2, we show the Schmidt decomposition of the spectrum. We use a moderate superradiant decay constant $N\bar{\mu} + 1 = 5$, comparable to the reference [9], and a nanosecond pulse duration $\tau = 0.25$ (in units of $1/\Gamma \approx 26$ ns), and $\Gamma_3/2\pi = 6$ MHz. Due to slow convergence associated with the Lorentzian profile, we use a frequency range up to ± 1200 (in unit of Γ_3) with 2000×2000 grid. The numerical error in the eigenvalue calculation is estimated to be about 1% error. In this case, the largest Schmidt number is 0.8 and corresponding signal mode function has a FWHM Gaussian profile $4\sqrt{2 \ln(2)}/\tau \approx 19\Gamma_3$. The idler mode function ϕ_1 reflects

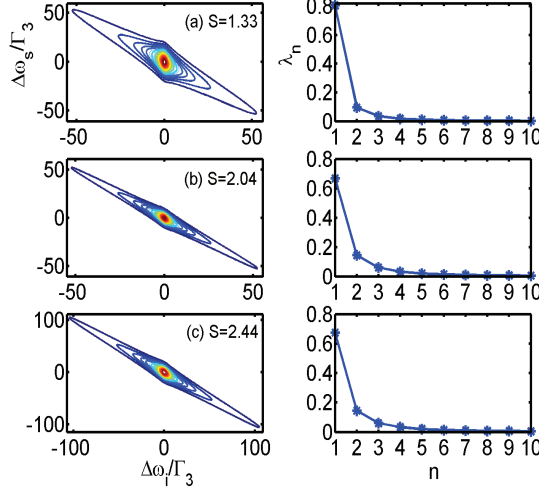


Figure 3. Absolute spectrum of two-photon state and the eigenvalues of Schmidt decomposition. $N\bar{\mu} + 1 = 5$ for both (a) $\tau = 0.25$ (b) $\tau = 0.5$. $N\bar{\mu} + 1 = 10$ for (c) $\tau = 0.25$. The von Neumann entropy (S) is indicated in the plots.

the Lorentzian profile in the spectrum at the signal peak frequency ($\Delta\omega_s = 0$),

$$f(\Delta\omega_s = 0, \Delta\omega_i) = \frac{e^{-\Delta\omega_i^2 \tau^2 / 8}}{(N\bar{\mu} + 1)\Gamma_3/2 - i\Delta\omega_i} \quad (21)$$

where a relatively broad Gaussian distribution is overlapped with a narrow spread of superradiant decay rate [FWHM $> (N\bar{\mu} + 1)\Gamma_3/2$].

Figure 3 shows that the cascade emission source is more entangled if the superradiant decay constant, or the pulse duration increases. We note that the Gaussian profile aligns the spectrum along the axis $\Delta\omega_s = -\Delta\omega_i$ and the spectral width for signal photon at the center of the idler frequency distribution ($\Delta\omega_i = 0$) is determined by pulse duration τ . For a shorter pulse $\tau^{-1} > (N\bar{\mu} + 1)\Gamma_3/2$, the joint Gaussian profile has a larger width, and the spectrum is cut off by the Lorentzian idler distribution. A larger width leads to a less entangled source and distributes the spectral weight mainly along the crossed axes $\Delta\omega_s = 0$ and $\Delta\omega_i = 0$. A narrow Lorentzian profile cuts off the entanglement source term $e^{-(\Delta\omega_s + \Delta\omega_i)^2 \tau^2 / 8}$ tilting the spectrum along the line $\Delta\omega_s + \Delta\omega_i = 0$. In the opposite limit, $\tau^{-1} < (N\bar{\mu} + 1)\Gamma_3/2$, the spectrum is highly entangled corresponding to tight alignment along the axis $\Delta\omega_s = -\Delta\omega_i$ (figure 3 (c)).

Note that the short pulse duration ($\tau \geq 0.25$ (6.5 ns)) should not violate the assumption of adiabaticity $\tau \gtrsim 1/\Delta_1$ or $1/\Delta_2$.

The Schmidt analysis and calculation of von Neumann entropy shows that signal-idler fields are more entangled if the ensemble is more optically dense, corresponding to stronger superradiance. For the DLCZ protocol, we wish to avoid frequency entanglement. The superradiance may be reduced with smaller atomic densities but good qubit storage and retrieval efficiency require a moderate optical thickness [9]. A better approach involves using short pulse excitation $\tau^{-1} > (N\bar{\mu} + 1)\Gamma_3$. We will investigate the spectral properties in more details for the DLCZ scheme in the next Section. Note that there has been a development in the setting of spontaneous

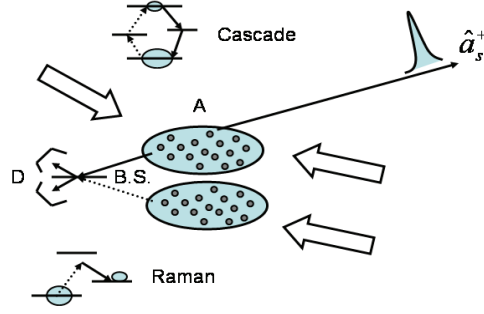


Figure 4. Entanglement generation in the DLCZ scheme using the cascade and Raman transitions in two different atomic ensembles. Large white arrows represent laser pump excitations corresponding to the dashed lines in either cascade or Raman level structures. Here \hat{a}_s^\dagger represents the emitted telecom photon. B.S. means beam splitter that is used to interfere the incoming photons measured by the photon detector D. The label A refers to the pair of ensembles for later reference.

parametric downconversion to generate frequency-uncorrelated entangled photons by using shorter pump pulses for scalable all-optical quantum information processing [31].

4. DLCZ scheme with cascade emission

In the DLCZ protocol, a weak pump laser Raman scatters a single photon generating a quantum correlated spin excitation in the ensemble. By interfering the Raman photons generated from two separate atomic ensembles on a beam splitter (B.S.), the DLCZ entangled state $(|01\rangle + |10\rangle)/\sqrt{2}$ [32] is prepared conditioned on one and only one click of the detectors after the B.S. Hence $|0\rangle$ and $|1\rangle$ represent the state of zero or one collective spin excitations stored in the hyperfine ground state coherences. This state originates from indistinguishable photon paths. The error from multiple excitations can be made negligible if the pump laser is weak enough.

As shown in figure 4, we consider instead that one of the ensembles employs cascade emission. The idea is for cascade emission to generate a telecom photon (\hat{a}_s^\dagger) for transmission in the optical fiber, and an infrared photon that interferes locally with the Raman photon generated in the Λ -type atomic ensemble. In this way interference of the infrared photons generate the entangled state,

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle_{a,s} + |10\rangle_{a,s}), \quad (22)$$

similar to the conventional DLCZ entanglement generation scheme. Now, however, instead of a stored spin excitation, we generate a telecom photon. Here we denote $|\Psi\rangle$ as a matter-light entangled state where (a, s) represent an atomic collective spin excitation and a telecom photon respectively.

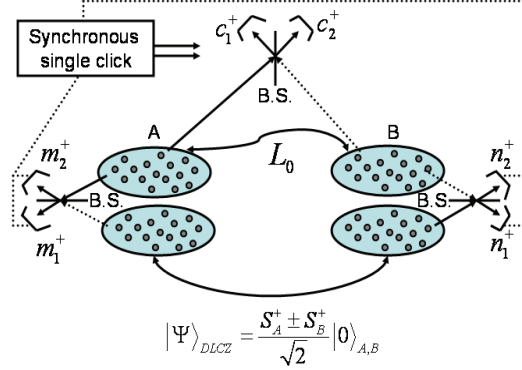


Figure 5. Entanglement swapping of DLCZ scheme using the cascade transition. The site A is described in detail in figure 4 and equivalently for the site B. The telecom signal photons are sent from both sites and interfere by B.S. midway between with detectors represented by c_1^\dagger and c_2^\dagger . Synchronous single clicks of the detectors from both sites ($m_{1,2}^\dagger, n_{1,2}^\dagger$) and the midway detector ($c_{1,2}^\dagger$) generate the entangled state between lower atomic ensembles at sites A and B. The locally generated entanglement is swapped to distantly separated sites in this cascade-emission-based DLCZ protocol.

The entanglement swapping with the cascade emission may be implemented as shown in figure 5, and will be discussed in detail in the next Section. The initial state is a tensor product of two state vectors generated locally at the sites A and B.

$$|\Psi\rangle_{AB} = (\sqrt{1-\eta_{1A}}|0\rangle + \sqrt{\eta_{1A}}|1\rangle_i^A|1\rangle_s^A) \otimes (\sqrt{1-\eta_{2A}}|0\rangle + \sqrt{\eta_{2A}}|1\rangle_r^A|1\rangle_a^A) \otimes (\sqrt{1-\eta_{1B}}|0\rangle + \sqrt{\eta_{1B}}|1\rangle_i^B|1\rangle_s^B) \otimes (\sqrt{1-\eta_{2B}}|0\rangle + \sqrt{\eta_{2B}}|1\rangle_r^B|1\rangle_a^B), \quad (23)$$

where (s, i) represent the signal and idler photons from the cascade emission, and (r, a) are Raman scattered photon and the collective spin excitation. Here η_1 and η_2 are efficiencies to generate cascade and Raman emission. Since η_1 and $\eta_2 \ll 1$, multiple atomic excitations or multi-photon generation can be excluded.

4.1. Entanglement swapping

Before we proceed to expand the product state of equation (23) and investigate the spectral effects of cascade emission on the modified DLCZ scheme, we would like to address the intrinsic errors from the protocols. Consider the product state generated from entangled states of A and B as in figure 5,

$$|\Psi\rangle_A \otimes |\Psi\rangle_B = \left(\frac{|10\rangle_{as} + |01\rangle_{as}}{\sqrt{2}} \right)_A \otimes \left(\frac{|10\rangle_{as} + |01\rangle_{as}}{\sqrt{2}} \right)_B \\ = \frac{1}{2}(|1010\rangle_{asas} + |1001\rangle_{asas} + |0110\rangle_{asas} + |0101\rangle_{asas}), \quad (24)$$

where the subscript (a) represents a stored local atomic excitation, and (s) means a telecom photon propagating toward the B.S. in the middle. We can tell from this

effective state that the first component ($|1010\rangle_{asas}$) contributes no telecom photons at all (two local excitations) and can be ruled out by measuring a "click" at one of the middle detectors. The second and the third components have components of the entangled state of quantum swapping, and the fourth one is the source of error if the photodetector cannot resolve one from two photons. The error could be corrected by using a photon number resolving detector (PNRD) if other drawbacks like dark counts, photon losses during propagation, and detector inefficiency are not considered.

Now we will formulate the entanglement swapping including the spectral effects discussed in Section 3. We ignore pump-phase offsets, assuming 50/50 B.S. and a symmetric set-up ($\eta_{1A} = \eta_{1B} = \eta_1$, $\eta_{2A} = \eta_{2B} = \eta_2$) for simplicity. Expand the previous joint state, equation (23) and keep the terms up to the second order of $\eta_{1,2}$ that can contribute to detection events ($\hat{m}_{1,2}^\dagger, \hat{n}_{1,2}^\dagger$),

$$\begin{aligned} |\Psi\rangle_{eff} = & \eta_1(1 - \eta_2)|1\rangle_i^A|1\rangle_s^A|1\rangle_i^B|1\rangle_s^B + \eta_2(1 - \eta_1)|1\rangle_r^A|1\rangle_{cs}^A|1\rangle_r^B|1\rangle_{cs}^B \\ & + \sqrt{\eta_1\eta_2(1 - \eta_1)(1 - \eta_2)}|1\rangle_i^A|1\rangle_s^A|1\rangle_r^B|1\rangle_{cs}^B \\ & + \sqrt{\eta_1\eta_2(1 - \eta_1)(1 - \eta_2)}|1\rangle_r^A|1\rangle_{cs}^A|1\rangle_i^B|1\rangle_s^B, \end{aligned} \quad (25)$$

where the cascade emission state $|1\rangle_s|1\rangle_i \equiv \int f(\omega_s, \omega_i) \hat{a}_{\lambda_s}^\dagger(\omega_s) \hat{a}_{\lambda_i}^\dagger(\omega_i) |0\rangle d\omega_s d\omega_i$ has the spectral distribution $f(\omega_s, \omega_i)$ as derived in Section 3.

As shown in figure 5, entanglement swapping protocol is fulfilled by measuring three clicks from the three pairs of the detectors respectively ($\hat{m}_{1,2}^\dagger, \hat{n}_{1,2}^\dagger, \hat{c}_{1,2}^\dagger$). The quantum efficiency of the detector is considered in the protocol, and we describe a model for quantum efficiency in Appendix B.1. We then use this model to describe photodetection events registered by non-resolving photon detectors (NRPD). Starting with the input density operator $\hat{\rho}_{in} = |\Psi\rangle_{eff}\langle\Psi|$, we derive the projected density operator, equation (B.16), conditioned on the three clicks of $\hat{m}_1^\dagger, \hat{n}_1^\dagger$, and \hat{c}_1^\dagger in Appendix C.2. We use the Schmidt decomposition of the projected density operator and assume a single mode for the Raman scattered photon. We find the un-normalized density operator $\hat{\rho}_{out}^{(2)}$ given in equation (B.16),

$$\begin{aligned} \hat{\rho}_{out}^{(2)} = & \frac{\eta_1^2(1 - \eta_2)^2}{16}(2 - \eta_t)\eta_t\eta_{eff}^2\left(1 + \sum_j \lambda_j^2\right)|0\rangle\langle 0| + \frac{\eta_1\eta_2(1 - \eta_1)(1 - \eta_2)}{8}\eta_t\eta_{eff}^2 \\ & \left\{ \left(\hat{S}_B^\dagger|0\rangle\langle 0|\hat{S}_B + \hat{S}_A^\dagger|0\rangle\langle 0|\hat{S}_A \right) + \sum_j \lambda_j \int \phi_j(\omega_i)\phi_j^*(\omega'_i)\Phi^*(\omega_i)\Phi^*(\omega'_i)d\omega_i d\omega'_i \right. \\ & \left. \left(\hat{S}_B^\dagger|0\rangle\langle 0|\hat{S}_A + \hat{S}_A^\dagger|0\rangle\langle 0|\hat{S}_B \right) \right\}, \end{aligned} \quad (26)$$

where η_t and η_{eff} are quantum efficiencies of the detectors at the telecom and infrared wavelengths respectively. λ_j 's are Schmidt eigenvalues derived in section 3. The first term in equation (26) is the atomic vacuum state at sites A and B and contributes an error to the output density operator. The second term contains the components of the DLCZ entangled state.

We can define the fidelity F as the projection of density operator to the entangled state $|\Psi\rangle_{DLCZ} = (S_A^\dagger + S_B^\dagger)|0\rangle/\sqrt{2}$ and calculate the success probability P_S of entanglement swapping of the entangled state and the heralding probability P_H for the third click as [33]

$$F \equiv \frac{\text{Tr}(\hat{\rho}_{out}^{(2)}|\Psi\rangle_{DLCZ}\langle\Psi|)}{\text{Tr}(\hat{\rho}_{out}^{(2)})}, \quad (27)$$

$$P_H = P_1 + P_2, \quad P_1 = P_2 = \frac{\text{Tr}(\hat{\rho}_{out}^{(2)})}{\mathcal{N}}, \quad (28)$$

$$P_S = P_1 \times F_1 + P_2 \times F_2, \quad F_1 = F_2 = F, \quad (29)$$

where $P_{1,2}$ is the heralding probability of the single click from the midway detector ($\hat{c}_{1,2}^\dagger$) as shown in figure 5, and a trace (Tr) is taken over atomic degrees of freedom. The normalization factor \mathcal{N} is calculated in equation (B.9) and is given by

$$\mathcal{N} = \frac{\eta_1^2(1-\eta_2)^2}{4}\eta_{eff}^2 + \frac{\eta_1\eta_2(1-\eta_1)(1-\eta_2)}{2}\eta_{eff}^2 + \frac{\eta_2^2(1-\eta_1)^2}{4}\eta_{eff}^2. \quad (30)$$

We have used the following properties for the calculation of $\hat{\rho}_{out}^{(2)}$ and \mathcal{N} ,

$$\int d\omega_s d\omega_i |f(\omega_s, \omega_i)|^2 = 1, \quad (31)$$

where orthonormal relations in the mode functions are used, and

$$\int d\omega_s d\omega'_s d\omega_i d\omega'_i f(\omega'_s, \omega'_i) f^*(\omega'_s, \omega_i) f(\omega_s, \omega_i) f^*(\omega_s, \omega'_i) = \sum_j \lambda_j^2. \quad (32)$$

Note that the single mode spectral function for the Raman photon satisfies $\int d\omega |\Phi(\omega)|^2 = 1$.

The fidelity, heralding, and success probability become

$$F = \frac{1 + \sum_j \lambda_j \int \phi_j(\omega_i) \phi_j^*(\omega'_i) \Phi^*(\omega_i) \Phi^*(\omega'_i) d\omega_i d\omega'_i}{\eta_r(2-\eta_t)(1 + \sum_j \lambda_j^2)/2 + 2}, \quad (33)$$

$$P_H = \frac{\eta_r \eta_t (2-\eta_t)(1 + \sum_j \lambda_j^2)/2 + 2\eta_t}{(\sqrt{\eta_r} + 1/\sqrt{\eta_r})^2}, \quad (34)$$

$$P_S = \eta_t \frac{1 + \sum_j \lambda_j \int \phi_j(\omega_i) \phi_j^*(\omega'_i) \Phi^*(\omega_i) \Phi^*(\omega'_i) d\omega_i d\omega'_i}{(\sqrt{\eta_r} + 1/\sqrt{\eta_r})^2}, \quad (35)$$

where $\frac{1-\eta_2}{1-\eta_1} \approx 1$ and $\eta_r = \eta_1/\eta_2$.

The fidelity depends on a sum of square of Schmidt numbers in the denominator and the mode mismatch between the idler and Raman photons in the numerator. Let us assume that the Raman photon mode is engineered to be matched with the idler photon mode of the largest Schmidt number ($\phi_1(\omega_i)$ in our case), which is required to have a larger fidelity (so is the success probability) compared to other modes. We may also compare the NRPD with the performance of PNRD in the midway detectors, then we have the fidelity, heralding, and success probability,

$$F = \begin{cases} \frac{1+\lambda_1}{\eta_r(2-\eta_t)(1+\sum_j \lambda_j^2)/2+2}, & \text{NRPD} \\ \frac{1+\lambda_1}{\eta_r(1-\eta_t)(1+\sum_j \lambda_j^2)+2}, & \text{PRND} \end{cases} \quad (36)$$

$$P_H = \begin{cases} \frac{\eta_r \eta_t (2-\eta_t)(1+\sum_j \lambda_j^2)/2+2\eta_t}{(\sqrt{\eta_r}+1/\sqrt{\eta_r})^2}, & \text{NRPD} \\ \frac{\eta_r \eta_t (1-\eta_t)(1+\sum_j \lambda_j^2)+2\eta_t}{(\sqrt{\eta_r}+1/\sqrt{\eta_r})^2}, & \text{PRND} \end{cases} \quad (37)$$

$$P_S = \begin{cases} \frac{\eta_t(1+\lambda_1)}{(\sqrt{\eta_r}+1/\sqrt{\eta_r})^2}, & \text{NRPD} \\ \frac{\eta_t(1+\lambda_1)}{(\sqrt{\eta_r}+1/\sqrt{\eta_r})^2}, & \text{PRND} \end{cases} \quad (38)$$

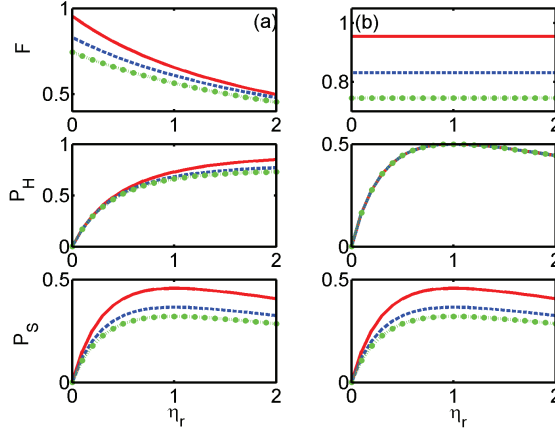


Figure 6. Fidelity F , heralding P_H , and success P_S probabilities of entanglement swapping versus relative efficiency η_r with perfect detection efficiency $\eta_t = 1$. Column (a) NRPD and (b) PNRD. Solid-red, dashed-blue, and dotted-green curves correspond to the pulse width parameters $\tau = (0.1, 0.5, 10)$ and superradiant factor $N\bar{\mu} + 1 = (5, 5, 10)$ (see Section 3 and Appendix A). The von Neumann entropy is $S = (0.684, 2.041, 2.886)$, respectively.

When the relative efficiency is made arbitrarily small, the fidelity approaches $(1 + \lambda_1)/2$ for both types of detectors. It reaches one if a pure cascade emission source is generated (von Neumann entropy $E = 0$ and $\lambda_1 = 1$). When $\eta_r = 1$ with a pure source using NRPD with a perfect quantum efficiency, $F = 2/3$, $P_H = 3/4$, $P_S = 1/2$, which coincide with the results of the reference [33] (with perfect quantum efficiency).

We discuss the frequency entanglement for various pulse widths and superradiant decay rates in Section 3. We find that for shorter driving pulses and smaller superradiant decay rates, the cascade emission source is less spectrally entangled. That means when η_r is fixed, a shorter driving pulse heralds a higher fidelity DLCZ entangled state.

In figure 6, we numerically calculate the entropy and plot out the fidelity from equation (36), the heralding probability from equation (37), and the success probability from equation (38) as a function of the relative efficiency η_r . With a perfect detection efficiency ($\eta_t = 1$), we find that at a smaller η_r , the less entangled source gives us a higher fidelity DLCZ entangled state but with a smaller success probability. Small generation probability for cascade emission ($\eta_r < 1$) reduces the error of NRPD from two telecom photons interference, but it reduces the successful entanglement swapping at the same time.

The optimal success probability occurs by using the same excitation efficiency for both cascade and Raman configurations. For PNRD, the fidelity is higher than NRPD, and the heralding probability is the same independent of the degree of frequency space entanglement. The success probabilities for both types of detectors are equal. The advantage of PNRD shows up in the fidelity of quantum swapping.

In figure 7, we show that the measures improve monotonically with the quantum efficiency ($\eta = \eta_t$) of the detector at telecom wavelength, with $\eta_r = 0.5$. The success

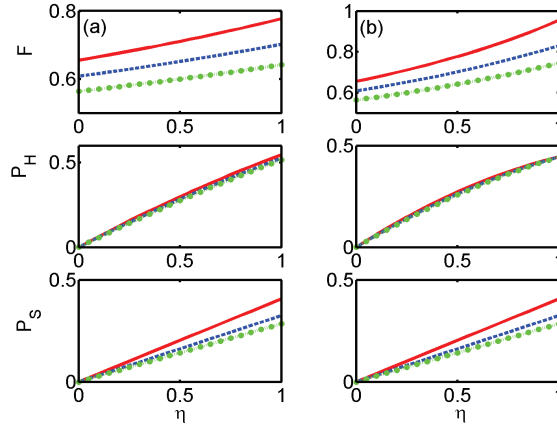


Figure 7. Fidelity F , heralding P_H , and success P_S probabilities of entanglement swapping versus telecom detector quantum efficiency η for the case of (a) NRPD and (b) PNRD. Solid-red, dashed-blue, and dotted-green curves correspond to the same parameters used in figure 6.

probabilities for both types of detectors are the same and again the advantage of PNRD shows up in the fidelity.

4.2. Effective ‘polarization’ maximally entangled (PME) state and quantum teleportation

In figure 8, we illustrate a scheme for probabilistic and effective PME state preparation and quantum teleportation. The term of ‘polarization’ is used as an analogy [3] to the entangled photons in polarization degree of freedom and note that what actually prepared here is the entangled photons in path modes. Four ensembles (ABCD) are used to generate two entangled pairs of DLCZ entangled states, and another two ensembles (I_1 , I_2) are used to prepare a quantum state to be teleported.

With the conditional output density matrix from equation (B.16), we proceed to construct the PME state $|\Psi\rangle_{PME} = \frac{1}{\sqrt{2}}(\hat{S}_A^\dagger \hat{S}_D^\dagger + \hat{S}_B^\dagger \hat{S}_C^\dagger)|0\rangle$ where (C, D) represents another parallel entanglement connection setup, figure 8 (a). This PME state is useful in entanglement-based communication schemes [3], and we will here calculate its success probability. The normalized density matrix for the AB system is from equation (26) (let $\eta_t = \eta$),

$$\begin{aligned} \hat{\rho}_{out,n}^{(2),AB} = & \frac{a}{a+4}|0\rangle\langle 0| + \frac{2}{a+4}\left(\hat{S}_B^\dagger|0\rangle\langle 0|\hat{S}_B + \hat{S}_A^\dagger|0\rangle\langle 0|\hat{S}_A \right. \\ & \left. + \lambda_1 \hat{S}_B^\dagger|0\rangle\langle 0|\hat{S}_A + \lambda_1 \hat{S}_A^\dagger|0\rangle\langle 0|\hat{S}_B\right), \end{aligned} \quad (39)$$

where the largest Schmidt number (λ_1) of mode overlap is chosen and $a \equiv \eta_r(2 - \eta) \left(1 + \sum_j \lambda_j^2\right)$.

A parallel pair of entangled ensembles (C,D) is introduced, and the joint density

operator is $\hat{\rho}_{out,n}^{(2),AB} \otimes \hat{\rho}_{out,n}^{(2),CD}$. The latter expression is developed mathematically in Appendix B.3.

With projection of the PME state, we have the post measurement success probability [a click from each side; the side of (A or C) and (B or D)],

$$\begin{aligned} P_{S,PME} &= \langle \Psi | \hat{\rho}_{out,n}^{(2),AB} \otimes \hat{\rho}_{out,n}^{(2),CD} | \Psi \rangle_{PME}, \\ &= \frac{4(1 + \lambda_1^2)}{[\eta_r(2 - \eta_t)(1 + \sum_j \lambda_j^2) + 4]^2}. \end{aligned} \quad (40)$$

For $\eta_r \ll 1$, $P_{S,PME}$ reaches the maximum of $1/2$ when a pure source ($\lambda_1 = 1$) is used. Compare with the original DLCZ proposal [3] where the success probability is $1/[2(c_0 + 1)^2]$ with vacuum coefficient c_0 in the entanglement generation, we have an equivalent form if a pure source is used, $P_{S,PME} = 1/[2(c_0 + 1)^2]$ where the vacuum coefficient of $\hat{\rho}_{out,n}^{(2),AB}$ can be expressed as $c_0 = \eta_r(2 - \eta)/2 = a/4$. We may use the PME state to enable the quantum cryptography and Bell inequality measurement by applying phases ϕ_L and ϕ_R to sides (A, C) and (B, D) in figure 8(a) respectively through single-bit operations [3]. For cascade-emission-based quantum communications, the spectral effect of the cascade emission we implement here reduces the success rate for generation of PME state because of frequency entanglement in the source where $\lambda_1 < 1$.

For an arbitrary quantum state transfer to long distance, quantum teleportation scheme may be used. Another two ensembles (I_1, I_2) are introduced [3], and the quantum state can be described by $|\Psi\rangle = (d_0 \hat{S}_{I_1}^\dagger + d_1 \hat{S}_{I_2}^\dagger)|0\rangle$ with $|d_0|^2 + |d_1|^2 = 1$.

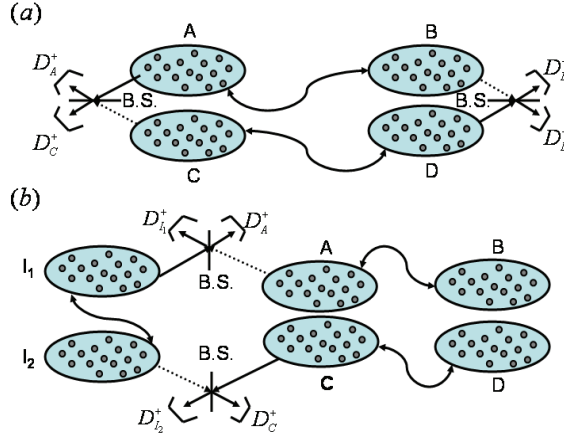


Figure 8. Effective PME projection (a) and quantum teleportation (b) in the DLCZ scheme. Four atomic ensembles (A,B,C,D) are used to generate two DLCZ entangled states at (A,B) and (C,D). PME state is projected probabilistically conditioned on four possible detection events of (D_A^\dagger or D_C^\dagger) and (D_B^\dagger or D_D^\dagger) in (a). In the quantum teleportation protocol (b), another two ensembles (I_1, I_2) are used to prepare a quantum state that is teleported to atomic ensembles B and D conditioned on four possible detection events of (\hat{D}_{I_1} or \hat{D}_A) and (\hat{D}_{I_2} or \hat{D}_C).

The joint density matrix for quantum teleportation is

$$\hat{\rho}_{QT} = (d_0 \hat{S}_{I_1}^\dagger + d_1 \hat{S}_{I_2}^\dagger) |0\rangle\langle 0| (d_0^* \hat{S}_{I_1} + d_1^* \hat{S}_{I_2}) \otimes \hat{\rho}_{out,n}^{(2),AB} \otimes \hat{\rho}_{out,n}^{(2),CD}. \quad (41)$$

Atomic ensembles (A,B) in parallel with (C,D) provide a scheme for PME state preparation. Retrieve the quantum state [ensemble (I_1, I_2)] into photons and interfere them at B.S., respectively, with photons from A and C. We have the teleported quantum state at B and D conditioned on the single click of $(\hat{D}_{I_1}$ or $\hat{D}_A)$ and $(\hat{D}_{I_2}$ or $\hat{D}_C)$.

Consider single detection events at \hat{D}_{I_1} and \hat{D}_{I_2} as an example. With the NRPD measurement operators $\hat{M}_{I_1, I_2} \equiv (\hat{I}_{D_1}^\dagger - |0\rangle_{D_1}\langle 0|) \otimes |0\rangle_{D_A}\langle 0| \otimes (\hat{I}_{D_2}^\dagger - |0\rangle_{D_2}\langle 0|) \otimes |0\rangle_{D_C}\langle 0|$ (we use D_1, D_2 for D_{I_1}, D_{I_2}), the density matrix after the measurement becomes

$$\begin{aligned} \hat{\rho}_1 \equiv \text{Tr}(\hat{\rho}_{QT,eff} \hat{M}_{I_1, I_2}) = \\ \frac{a+2}{2(a+4)^2} |0\rangle_{ABCD} \langle 0| + \frac{4}{(a+4)^2} \left(\frac{|d_0|^2}{4} \hat{S}_B^\dagger |0\rangle\langle 0| \hat{S}_B + \frac{|d_1|^2}{4} \hat{S}_D^\dagger |0\rangle\langle 0| \hat{S}_D + \right. \\ \left. \frac{\lambda_1^2 d_0 d_1^*}{4} \hat{S}_B^\dagger |0\rangle\langle 0| \hat{S}_D + \frac{\lambda_1^2 d_0^* d_1}{4} \hat{S}_D^\dagger |0\rangle\langle 0| \hat{S}_B \right), \end{aligned} \quad (42)$$

where $\hat{\rho}_{QT,eff}$ is calculated in equation (B.18), and the trace is taken over the electromagnetic field degrees of freedom.

For a successful transfer of the quantum state $|\Phi\rangle = (d_0 \hat{S}_B^\dagger + d_1 \hat{S}_D^\dagger) |0\rangle$, the fidelity $F_1 = \langle \Phi | \hat{\rho}_1 | \Phi \rangle / \text{Tr}(\hat{\rho}_1)$, and the heralding probability is $P_1 = \text{Tr}(\hat{\rho}_1)$, with the trace over all atomic degrees of freedom. Except for the detection event we consider here, there are three other detection events including (D_A, D_C) , (D_{I_1}, D_C) and (D_A, D_{I_2}) . The teleported state from the detection events (D_{I_1}, D_C) and (D_{I_2}, D_A) requires a π rotation correction on the relative phase ($d_0 \rightarrow d_0, d_1 \rightarrow -d_1$).

The fidelity and heralding probabilities conditioned on the other three pairs of clicks are the same as F_1 and P_1 respectively, so the success probability is

$$\begin{aligned} P_{S,QT} &= \sum_i^4 P_i F_i = 4P_1 F_1, \\ &= \frac{F^2}{(1+\lambda_1)^2} [1 + (2\lambda_1^2 - 2)|d_0|^2 |d_1|^2], \end{aligned} \quad (43)$$

where F is the fidelity of entanglement swapping for NRPD, equation (36). For PNRD, the success probability for quantum teleportation is unchanged.

The success probability for quantum teleportation depends on the probability amplitude of the quantum state and the fidelity F of the entanglement swapping. In figure 9, for $\eta_r = 0.5$ and $\eta_t = 1$, we can see in the region $|d_0| \approx 0.3 \sim 0.9$, higher success probability requires a less entangled cascade emission source. Outside this region, it prefers a more entangled source. When a pure source is used ($\lambda_1 = 1$) and let $\eta_r \ll 1, \eta_t = 1$, we can achieve the maximum of the success probability $P_{S,QT} = \frac{1}{4}$ when $F = 1$, which is also achieved in the traditional DLCZ scheme with perfect quantum efficiencies [33].

5. Discussions and conclusions

We have described probabilistic protocols for the DLCZ scheme implementing the cascade emission source. We characterize the spectral properties of the cascade

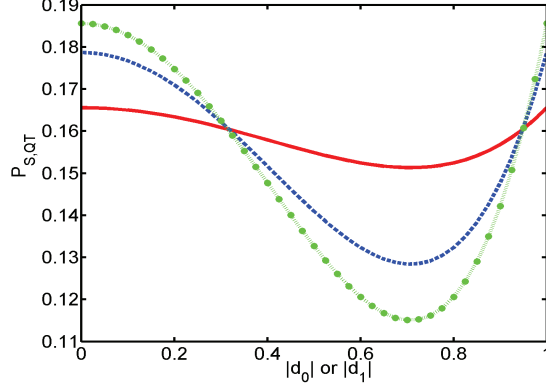


Figure 9. Success probability of quantum teleportation as a function of the probability amplitude of teleported quantum state with $\eta_r = 0.5$ and a perfect detector efficiency $\eta_t = 1$. Solid-red, dashed-blue, and dotted-green curves correspond to the same parameters used in figure 6.

emission by Schmidt mode analysis and investigate the fidelity and success probability of the protocols using photon resolving and non-resolving photon detectors. The success probability is independent of the detector type, but photon number resolving detection improves the fidelity.

The performance of the protocol also depends on the ratio of efficiencies in generating the cascade and Raman photons. The success probability is optimized for equal efficiencies while the fidelity is higher when the ratio is smaller than one for non-resolving photon detectors.

The frequency space entanglement of telecom photons produced in cascade emission deteriorates the performance of DLCZ protocols. The harmful effect can be diminished by using shorter pump pulses to generate the cascade emission. A state dependent success probability of quantum teleportation was calculated, and in some cases a more highly frequency entangled cascade emission source teleports more successfully. An improved performance could be achieved if the error source (vacuum part) were removed. This could be done by entanglement purification [26] at the stage of entanglement swapping and then using the purified source to teleport the quantum state.

The quantum efficiency of detectors have improved to above 60% in infra-red wavelength for avalanche photodiodes (APDs) and a maximum of 95% in telecom wavelength for superconducting devices at very low temperature (100 mK) [34]. We expect our optimal performance in the modified DLCZ scheme can be achieved as shown in figure (7) where a fidelity $F \approx 0.9$. Our cascade-emission-based quantum communication scheme utilizes a telecom wavelength photon that has minimal loss 0.2 dB/km through fiber transmission. Compare with 2 dB/km loss for infra-red bandwidth, telecom photon has an attenuation length ten times longer which is about 22 km. In terms of the rate of direct single photon transmission over continental distances (several hundreds kilometers), it is overwhelmingly desirable to use telecom

over infra-red bandwidth [34].

We note that an alternative method to generate telecom photons in atomic ensembles is frequency down conversion [10]. Two cold and non-degenerate rubidium gas samples are used to correlate a stored atomic excitation and a telecom photon. The stored excitation is correlated with an infra-red photon (idler) in one sample, and the idler is converted to a telecom wavelength photon in the other ensemble. Thus a matter-light entanglement is created to serve as a basic element in entanglement connection of DLCZ scheme with an advantageous telecommunication bandwidth. Similar to our cascade emission scheme, frequency conversion also requires a phase-matching of four-wave mixing condition in a diamond configuration of atomic levels [9, 35]. To implement it into our modified DLCZ scheme, an extra conversion efficiency needs to be taken into account. The efficiency has reached a maximum of 0.54 [10] and can be close to one if we use atoms with larger optical depth ($\text{opd} > 200$) [11]. Therefore this alternative method serves as well as our cascade emission scheme but demands one more cold atomic ensemble which might cause difficulties when large scale quantum repeater is considered.

Acknowledgments

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Appendix A. Hamiltonian and Schrödinger Equation

In this appendix, we derive the Hamiltonian for the cascade emission (signal-idler) from a four-level atomic ensemble. We use Schrödinger's equation to study the correlated two-photon state from a two-photon laser excitation. Consider an ensemble of N four-level atoms interacting with two classical fields and spontaneously emitted signal and idler photons as shown in figure 1. These identical atoms distribute randomly with a uniform density. Use dipole approximation of light-matter interactions, $-\vec{d} \cdot \vec{E}$ where \vec{E} is classical or quantum electric field, and rotating wave approximation (RWA) [29], the Hamiltonian in interaction picture is

$$\begin{aligned} V_I(t) = & -\hbar\Delta_1 \sum_{\mu=1}^N |1\rangle_{\mu}\langle 1| - \hbar\Delta_2 \sum_{\mu=1}^N |2\rangle_{\mu}\langle 2| - \frac{\hbar}{2} \left\{ \Omega_a \hat{P}_{1\vec{k}_a}^{\dagger} + \Omega_b \hat{P}_{2\vec{k}_b}^{\dagger} + h.c. \right\} \\ & - i\hbar \left\{ \sum_{\vec{k}_s, \lambda_s} g_{k_s} (\epsilon_{\vec{k}_s, \lambda_s} \cdot \hat{d}_s^*) \hat{a}_{\vec{k}_s, \lambda_s} \hat{S}_{\vec{k}_s}^{\dagger} e^{-i(\omega_{k_s} - \omega_{23} - \Delta_2)t} \right. \\ & \left. + \sum_{\vec{k}_i, \lambda_i} g_{k_i} (\epsilon_{\vec{k}_i, \lambda_i} \cdot \hat{d}_i^*) \hat{a}_{\vec{k}_i, \lambda_i} \hat{I}_{\vec{k}_i}^{\dagger} e^{-i(\omega_{k_i} - \omega_3)t} - h.c. \right\}, \end{aligned} \quad (\text{A.1})$$

where the collective dipole operators, and positive frequency parts of the electric fields are defined as

$$\begin{aligned} \hat{P}_{1\vec{k}_a}^{\dagger} & \equiv \sum_{\mu} |1\rangle_{\mu}\langle 0| e^{i\vec{k}_a \cdot \vec{r}_{\mu}}, \quad \hat{P}_{2\vec{k}_b}^{\dagger} \equiv \sum_{\mu} |2\rangle_{\mu}\langle 1| e^{i\vec{k}_b \cdot \vec{r}_{\mu}}, \\ \hat{S}_{\vec{k}_s}^{\dagger} & \equiv \sum_{\mu} |2\rangle_{\mu}\langle 3| e^{i\vec{k}_s \cdot \vec{r}_{\mu}}, \quad \hat{I}_{\vec{k}_i}^{\dagger} \equiv \sum_{\mu} |3\rangle_{\mu}\langle 0| e^{i\vec{k}_i \cdot \vec{r}_{\mu}}, \end{aligned} \quad (\text{A.2})$$

$$\begin{aligned}\hat{E}_s^+(\vec{r}_1, t_1) &= \sum_{k_s, \lambda_s} \sqrt{\frac{\hbar \omega_s}{2\epsilon_0 V}} \hat{a}_{k_s, \lambda_s} \vec{\epsilon}_{k_s, \lambda_s} e^{i\vec{k}_s \cdot \vec{r}_1 - i\omega_s t_1}, \\ \hat{E}_i^+(\vec{r}_2, t_2) &= \sum_{k_i, \lambda_i} \sqrt{\frac{\hbar \omega_i}{2\epsilon_0 V}} \hat{a}_{k_i, \lambda_i} \vec{\epsilon}_{k_i, \lambda_i} e^{i\vec{k}_i \cdot \vec{r}_2 - i\omega_i t_2}.\end{aligned}\quad (\text{A.3})$$

The time dependence of optical frequency in driving fields are absorbed by signal and idler fields. Single photon detuning $\Delta_1 = \omega_a - \omega_1$, and two-photon detuning $\Delta_2 = \omega_a + \omega_b - \omega_2$, $\omega_{23} = \omega_2 - \omega_3$. Rabi frequencies are $\Omega_a \equiv (1|\hat{d}|0)\mathcal{E}(k_a)/\hbar$, $\Omega_b \equiv (2|\hat{d}|1)\mathcal{E}(k_b)/\hbar$ and coupling coefficients are $g_{ks} \equiv (3|\hat{d}|2)\mathcal{E}(k_s)/\hbar$, $g_{ki} \equiv (0|\hat{d}|3)\mathcal{E}(k_i)/\hbar$. The double matrix element of the dipole moment is independent of the hyperfine structure and $\mathcal{E}(k) = \sqrt{\frac{\hbar k c}{2\epsilon_0 V}}$. Polarizations of signal and idler fields $\epsilon_{k_s, \lambda_s}$, $\epsilon_{k_i, \lambda_i}$ and unit direction of dipole operators \hat{d}_s , \hat{d}_i .

In the limit of large detuned and weak driving fields which satisfy $\Delta_1 \gg \frac{\sqrt{N}|\Omega_a|}{2}$, we consider only single excitation and ignore spontaneous decay during excitation. The state function can be described by

$$\begin{aligned}|\psi(t)\rangle &= \mathcal{E}(t)|0, vac\rangle + \sum_{\mu=1}^N A_\mu(t)|1_\mu, vac\rangle + \sum_{\mu=1}^N B_\mu(t)|2_\mu, vac\rangle \\ &+ \sum_{\mu=1}^N \sum_{k_s, \lambda_s} C_s^\mu(t)|3_\mu, 1_{\vec{k}_s, \lambda_s}\rangle + \sum_{k_s, \lambda_s, k_i, \lambda_i} D_{s,i}(t)|0, 1_{\vec{k}_s, \lambda_s}, 1_{\vec{k}_i, \lambda_i}\rangle,\end{aligned}\quad (\text{A.4})$$

where $s = (k_s, \lambda_s)$, $i = (k_i, \lambda_i)$, $|m_\mu\rangle \equiv |m_\mu\rangle|0\rangle_{\nu \neq \mu}^{\otimes N-1}$, $m = 1, 2, 3$ and $|vac\rangle$ is the vacuum photon state. The probability amplitudes coupled from rotating wave terms in the Hamiltonian are $\mathcal{E}(t)$, $A_\mu(t)$, $B_\mu(t)$, $C_s^\mu(t)$, $D_{s,i}(t)$, which indicate the complete cycle of single excitation process from the ground state, intermediate, upper excited state, intermediate excited state with emission of a signal photon, and the ground state with the signal-idler emission. Apply Schrödinger equation $i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle = V_I(t)|\psi(t)\rangle$ and we have the coupled equations of motion,

$$i\dot{\mathcal{E}} = -\frac{\Omega_a^*}{2} \sum_{\mu} e^{-i\vec{k}_a \cdot \vec{r}_\mu} A_\mu, \quad (\text{A.5})$$

$$i\dot{A}_\mu = -\frac{\Omega_a}{2} e^{i\vec{k}_a \cdot \vec{r}_\mu} \mathcal{E} - \frac{\Omega_b^*}{2} e^{-i\vec{k}_b \cdot \vec{r}_\mu} B_\mu - \Delta_1 A_\mu, \quad (\text{A.6})$$

$$\begin{aligned}i\dot{B}_\mu &= -\frac{\Omega_b}{2} e^{i\vec{k}_b \cdot \vec{r}_\mu} A_\mu - \Delta_2 B_\mu \\ &- i \sum_{k_s, \lambda_s} g_{ks} (\epsilon_{k_s, \lambda_s}^* \cdot \hat{d}_s^*) e^{i\vec{k}_s \cdot \vec{r}_\mu} e^{-i(\omega_{ks} - \omega_{23} - \Delta_2)t} C_s^\mu,\end{aligned}\quad (\text{A.7})$$

$$\begin{aligned}\dot{C}_s^\mu &= ig_{ks}^* (\epsilon_{k_s, \lambda_s}^* \cdot \hat{d}_s) e^{-i\vec{k}_s \cdot \vec{r}_\mu} e^{i(\omega_{ks} - \omega_{23} - \Delta_2)t} B_\mu \\ &- i \sum_{k_i, \lambda_i} g_{ki} (\epsilon_{k_i, \lambda_i}^* \cdot \hat{d}_i^*) e^{i\vec{k}_i \cdot \vec{r}_\mu} e^{-i(\omega_{ki} - \omega_3)t} D_{s,i},\end{aligned}\quad (\text{A.8})$$

$$i\dot{D}_{s,i} = ig_{ki}^* (\epsilon_{k_i, \lambda_i}^* \cdot \hat{d}_i) \sum_{\mu} e^{-i\vec{k}_i \cdot \vec{r}_\mu} e^{i(\omega_{ki} - \omega_3)t} C_s^\mu. \quad (\text{A.9})$$

In the limit of large detunings,

$$|\Delta_1|, |\Delta_2| \gg \frac{|\Omega_a|}{2}, \frac{|\Omega_b|}{2}, \frac{\Gamma_2}{2},$$

where Γ_2 is the natural decay rate for the upper excited state. We can solve the coupled equations of motion by adiabatically eliminating the intermediate and upper excited states. The adiabatic approximation [29] requires the smoothly turned on of the driving pulses, and it is equivalent to solve for the steady state solutions of the above coupled equations in a perturbative manner. Since we have weak pump fields, the ground state is constant and other probability amplitudes are

$$A_\mu(t) \approx -\frac{\Omega_a(t)}{2\Delta_1} e^{i\vec{k}_a \cdot \vec{r}_\mu}, \quad (\text{A.10})$$

$$B_\mu(t) \approx \frac{\Omega_a(t)\Omega_b(t)}{4\Delta_1\Delta_2} e^{i(\vec{k}_a + \vec{k}_b) \cdot \vec{r}_\mu}, \quad (\text{A.11})$$

where probability amplitude of first excited state follows the first driving field and the upper excited state follows the multiplication of two driving fields.

Substitute equation (A.9) into equation (A.8), we have differential equation for probability amplitude $C_s^\mu(t)$,

$$\begin{aligned} \dot{C}_s^\mu(t) = & g_s^* (\epsilon_{k_s, \lambda_s}^* \cdot \hat{d}_s) e^{-i\vec{k}_s \cdot \vec{r}_\mu} e^{i(\omega_{k_s} - \omega_{23} - \Delta_2)t} B_\mu(t) \\ & - \sum_\nu \sum_{k_i, \lambda_i} |g_i|^2 |\epsilon_{k_i, \lambda_i} \cdot \hat{d}_i^*|^2 e^{i\vec{k}_i \cdot (\vec{r}_\mu - \vec{r}_\nu)} \int_0^t dt' e^{i(\omega_{k_i} - \omega_3)(t' - t)} C_s^\nu(t'). \end{aligned} \quad (\text{A.12})$$

Define a phased probability amplitude $C_{s, q_i} = \sum_\mu C_s^\mu e^{-i\vec{q}_i \cdot \vec{r}_\mu}$, substitute C_s^ν with $\frac{1}{N} \sum_{q'_i} C_{s, q'_i} e^{i\vec{q}'_i \cdot \vec{r}_\nu}$, and identify the terms of the summation of exponential factors, $\sum_\mu e^{i(\vec{k}_i - \vec{q}_i) \cdot \vec{r}_\mu}$ or $\sum_\nu e^{-i(\vec{k}_i - \vec{q}'_i) \cdot \vec{r}_\nu}$, the coupling from the modes \vec{q}_i and \vec{q}'_i are significant only when $|\vec{q}'_i| = |\vec{k}_i| = |\vec{q}_i|$, so finally we have

$$\dot{C}_{s, q_i} = g_s^* (\epsilon_s^* \cdot \hat{d}_s) \sum_\mu e^{-i(\vec{k}_s + \vec{q}_i) \cdot \vec{r}_\mu} e^{i(\omega_{k_s} - \omega_{23} - \Delta_2)t} B_\mu - \frac{\Gamma_3}{2} (N\bar{\mu} + 1) C_{s, q_i} + i\delta\omega_i C_{s, q_i}. \quad (\text{A.13})$$

The collective decay rate is [27, 36]

$$\frac{\Gamma_3}{2} (N\bar{\mu} + 1) \equiv \frac{\Gamma_3}{2} \frac{3}{8\pi} \oint d\Omega_i [1 - (\hat{k}_i \cdot \hat{d}_i)^2] \frac{1}{N} \sum_{\mu, \nu} e^{i(\vec{k}_i - \vec{q}_i) \cdot (\vec{r}_\mu - \vec{r}_\nu)}, \quad (\text{A.14})$$

and the collective frequency shift expressed in terms of the continuous integral over a frequency space is [27, 37, 38, 28]

$$\delta\omega_i \equiv \int_0^\infty d\omega_i \frac{\Gamma_i}{2\pi} [\text{P.V.}(\omega_i - \omega_3)^{-1} + \text{P.V.}(\omega_i + \omega_3)^{-1}] N\bar{\mu}(k_i), \quad (\text{A.15})$$

$$\bar{\mu}(k_i) = \frac{1}{N^2} \sum_{\mu, \nu \neq \mu} e^{i(\vec{k}_i - \vec{q}_i) \cdot (\vec{r}_\mu - \vec{r}_\nu)}, \quad (\text{A.16})$$

which is derived after we renormalize the Lamb shift and consider the non-RWA terms in the original Hamiltonian. Non-RWA terms contribute to the term proportional to $\text{P.V.}(\omega_i + \omega_3)^{-1}$.

The geometrical constant $\bar{\mu}$ for a cylindrical ensemble (of height h and radius a) is

$$\bar{\mu}(k_3) = \frac{6(N-1)}{NA^2H^2} \int_{-1}^1 \frac{dx(1+x^2)}{(1-x)^2(1-x^2)} \sin^2\left[\frac{1}{2}H(1-x)\right] J_1^2[A(1-x^2)^{1/2}], \quad (\text{A.17})$$

where $H = k_3h$ and $A = k_3a$ are dimensionless length scales, and circular polarizations are considered [36]. J_1 is the Bessel function of the first kind.

Appendix B. Multimode Description of Correlated Two-Photon State

In this Appendix, we review a general model for quantum detection efficiency [39] for multimode analysis in various quantum communication scheme. Based on this detection model with the spectral description of correlated two-photon state, we derive the effective density matrix conditioning on the detection events of entanglement swapping, polarization maximally entangled (PME) state projection, and quantum teleportation.

Appendix B.1. Quantum Efficiency of Detector

To account for quantum efficiency of detector and the affect of its own spectrum filtering, we introduce an extra beam splitter (B.S.) with a transmissivity $\eta(\omega, \omega_0)$ [39] before the detection event. η models the quantum efficiency of the detectors in the microscopic level (response at frequency ω_0) and the macroscopic level (time-integrated detection). One example of conditioning on the single click of the detector, the output density operator becomes

$$\hat{\rho}_{out} = \int_{-\infty}^{\infty} d\omega_0 \hat{\Pi}_1 \text{Tr}_{ref} [\hat{U}_{BS} \hat{\rho}_{in} \hat{U}_{BS}^\dagger] \hat{\Pi}_1, \quad (\text{B.1})$$

$$\hat{\Pi}_1 \equiv \int_{-\infty}^{\infty} d\omega |\omega\rangle \langle \omega|, \quad (\text{B.2})$$

$$\hat{U}_{BS} \equiv \begin{pmatrix} \sqrt{1-\eta} & \sqrt{\eta} \\ \sqrt{\eta} & -\sqrt{1-\eta} \end{pmatrix}, \quad (\text{B.3})$$

where Tr_{ref} is the trace over the reflected modes m_3^\dagger , and the flat spectrum projection operator $\hat{\Pi}_1$ (only photon number is projected and no frequency resolution) is considered in the measurement process [23]. In figure B1, m_1^\dagger is the incoming photon operator before the detection, m_3^\dagger is the reflected mode, and m_4^\dagger is now the detection mode with a modelling of spectral quantum efficiency and an effective quantum efficiency is defined as

$$\int_{-\infty}^{\infty} \eta(\omega, \omega_0) d\omega_0 = \eta_{eff}(\omega). \quad (\text{B.4})$$

Appendix B.2. Multimode Description of Entanglement Swapping

From equation (25), we use single mode $\Phi(\omega)$ for Raman photon and a multimode description $f(\omega_s, \omega_i)$ for cascade photons and rewrite the effective state. Note that a symmetric setup is considered so the mode description is the same for both sides A and B in the scheme of entanglement swapping.

$$\begin{aligned} |\Psi\rangle_{eff} = & \eta_1(1 - \eta_2) \times \\ & \int f(\omega_s, \omega_i) \hat{a}_s^{\dagger, A}(\omega_s) \hat{a}_i^{\dagger, A}(\omega_i) d\omega_s d\omega_i \int f(\omega'_s, \omega'_i) \hat{a}_s^{\dagger, B}(\omega'_s) \hat{a}_i^{\dagger, B}(\omega'_i) d\omega'_s d\omega'_i |0\rangle + \\ & \eta_2(1 - \eta_1) \int \Phi(\omega) d\omega \hat{a}_r^{\dagger, A}(\omega) \hat{S}_A^\dagger \int \Phi(\omega') d\omega' \hat{a}_r^{\dagger, B}(\omega') \hat{S}_B^\dagger |0\rangle + \sqrt{\eta_1(1 - \eta_1)} \times \end{aligned}$$

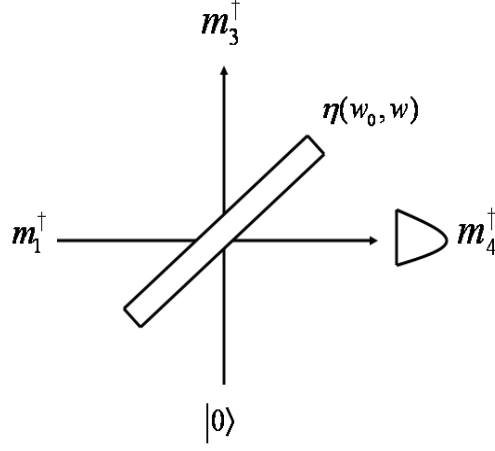


Figure B1. Model of quantum efficiency of detector.

$$\begin{aligned} & \sqrt{\eta_2(1-\eta_2)} \int f(\omega_s, \omega_i) d\omega_s d\omega_i \times \hat{a}_s^{\dagger,A}(\omega_s) \hat{a}_i^{\dagger,A}(\omega_i) \int \Phi(\omega') d\omega' \hat{a}_r^{\dagger,B}(\omega') \hat{S}_B^\dagger |0\rangle + \\ & \sqrt{\eta_1\eta_2(1-\eta_1)(1-\eta_2)} \int \Phi(\omega) d\omega \hat{a}_r^{\dagger,A}(\omega) \hat{S}_A^\dagger \int f(\omega'_s, \omega'_i) \hat{a}_s^{\dagger,B}(\omega'_s) \hat{a}_i^{\dagger,B}(\omega'_i) d\omega'_s d\omega'_i |0\rangle. \end{aligned} \quad (\text{B.5})$$

With the B.S., we have $\hat{a}_i^{\dagger,A} = \frac{\hat{m}_1^\dagger + \hat{m}_2^\dagger}{\sqrt{2}}$, $\hat{a}_i^{\dagger,B} = \frac{\hat{n}_1^\dagger + \hat{n}_2^\dagger}{\sqrt{2}}$, $\hat{a}_r^{\dagger,A} = \frac{\hat{m}_1^\dagger - \hat{m}_2^\dagger}{\sqrt{2}}$, $\hat{a}_r^{\dagger,B} = \frac{\hat{n}_1^\dagger - \hat{n}_2^\dagger}{\sqrt{2}}$, where \hat{a}_i^\dagger is the creation operator for idler photon and \hat{a}_r^\dagger is for Raman photon. The input density operator is $\hat{\rho}_{in} = |\Psi\rangle_{eff} \langle\Psi|$ and conditioning on the pair of single click $(\hat{m}_{1,2}^\dagger, \hat{n}_{1,2}^\dagger)$, we are able to generate maximally entangled singlet or triplet state $|\Psi\rangle_{DLCZ} = \frac{S_A^\dagger \pm S_B^\dagger}{\sqrt{2}} |0\rangle_{A,B}$. Without loss of generality, we consider a triplet state along with a pair of clicks $(\hat{m}_1^\dagger, \hat{n}_1^\dagger)$ and use the model of quantum efficiency in equation (B.1) with tracing over the detection modes $(\hat{m}_4^\dagger, \hat{n}_4^\dagger)$. Note that $\hat{m}_1^\dagger = \sqrt{1-\eta} \hat{m}_3^\dagger + \sqrt{\eta} \hat{m}_4^\dagger$ and $\hat{n}_1^\dagger = \sqrt{1-\eta} \hat{n}_3^\dagger + \sqrt{\eta} \hat{n}_4^\dagger$ as we model the quantum efficiency in the previous Section.

$$\hat{\rho}_{out} = \int_{-\infty}^{\infty} d\omega_0 \text{Tr}_{m4,n4} \{ \text{Tr}_{m3,n3} [\hat{U}_{BS}^B \hat{U}_{BS}^A \hat{\rho}_{in} \hat{U}_{BS}^{\dagger,A} \hat{U}_{BS}^{\dagger,B}] \hat{M}_{4,4} \}, \quad (\text{B.6})$$

$$\hat{M}_{4,4} \equiv (\hat{I}_{m4}^\dagger - |0\rangle_{m4} \langle 0|) \otimes |0\rangle_{m2} \langle 0| \otimes (\hat{I}_{n4}^\dagger - |0\rangle_{n4} \langle 0|) \otimes |0\rangle_{n2} \langle 0|, \quad (\text{B.7})$$

where the unitary B.S. operator is denoted by both sides (A and B) and NRPD projection operators are used [33]. These operators project the state with single click of the detected mode without resolving the number of photons. \hat{I} is identity operator. The un-normalized output density operator after tracing out these modes becomes

$$\begin{aligned}
\hat{\rho}_{out} = & \frac{\eta_1^2(1-\eta_2)^2}{4} \times \\
& \int d\omega_i d\omega'_i \eta_{eff}(\omega_i) \eta_{eff}(\omega'_i) \left[\int f(\omega_s, \omega_i) \hat{a}_s^{\dagger, A}(\omega_s) d\omega_s \int f(\omega'_s, \omega'_i) \hat{a}_s^{\dagger, B}(\omega'_s) d\omega'_s \right] \\
& |0\rangle\langle 0| \left[\int f^*(\omega''_s, \omega_i) \hat{a}_s^A(\omega''_s) d\omega''_s \int f^*(\omega'''_s, \omega'_i) \hat{a}_s^B(\omega'''_s) d\omega'''_s \right] \\
& + \frac{\eta_1 \eta_2 (1-\eta_1)(1-\eta_2)}{4} \left\{ \int d\omega_i \eta_{eff}(\omega_i) \left[\int f(\omega_s, \omega_i) d\omega_s \int f^*(\omega'_s, \omega_i) d\omega'_s \right. \right. \\
& \left. \left. \int |\Phi(\omega)|^2 \eta_{eff}(\omega) d\omega \right] \left(\hat{a}_s^{\dagger, A}(\omega_s) \hat{S}_B^\dagger |0\rangle\langle 0| \hat{S}_B \hat{a}_s^A(\omega'_s) + \right. \right. \\
& \left. \left. \hat{a}_s^{\dagger, B}(\omega_s) \hat{S}_A^\dagger |0\rangle\langle 0| \hat{S}_A \hat{a}_s^B(\omega'_s) \right) + \int \int f(\omega_s, \omega_i) d\omega_s \Phi^*(\omega_i) \eta_{eff}(\omega_i) d\omega_i \times \right. \\
& \left. \int \int f^*(\omega'_s, \omega'_i) d\omega'_s \Phi(\omega'_i) \eta_{eff}(\omega'_i) d\omega'_i \left(\hat{a}_s^{\dagger, A}(\omega_s) \hat{S}_B^\dagger |0\rangle\langle 0| \hat{S}_A \hat{a}_s^B(\omega'_s) + \right. \right. \\
& \left. \left. \hat{a}_s^{\dagger, B}(\omega_s) \hat{S}_A^\dagger |0\rangle\langle 0| \hat{S}_B \hat{a}_s^A(\omega'_s) \right) \right\} + \hat{\rho}'_{out}, \tag{B.8}
\end{aligned}$$

where $\eta_{eff}(\omega)$ is introduced after integration of ω_0 , and we denote it as an effective quantum efficiency for idler field ω_i or Raman photon at frequency ω (wavelength 780 nm for D2 line of Rb atom). $\hat{\rho}'_{out}$ includes the terms that won't survive after the interference of telecom photons in the middle B.S. (conditioning on a single click of detector). They involve operators like $\hat{a}_s^{\dagger, A} \hat{a}_s^{\dagger, B} |0\rangle\langle 0| \hat{a}_s^A \hat{S}_B$, $\hat{a}_s^{\dagger, A} \hat{a}_s^{\dagger, B} |0\rangle\langle 0| \hat{S}_A \hat{S}_B$ and $\hat{S}_B^{\dagger, A} \hat{S}_B^{\dagger, B} |0\rangle\langle 0| \hat{S}_A \hat{S}_B$.

The normalization factor is derived by tracing over the atomic degree of freedom.

$$\begin{aligned}
\text{Tr}(\hat{\rho}_{out}) \equiv \mathcal{N} = & \frac{\eta_1^2(1-\eta_2)^2}{4} \int d\omega_s d\omega_i \eta_{eff}(\omega_i) |f(\omega_s, \omega_i)|^2 \int d\omega'_s d\omega'_i \eta_{eff}(\omega'_i) |f(\omega'_s, \omega'_i)|^2 + \\
& \frac{\eta_1 \eta_2 (1-\eta_1)(1-\eta_2)}{2} \int d\omega_s d\omega_i \eta_{eff}(\omega_i) |f(\omega_s, \omega_i)|^2 \int |\Phi|^2(\omega) \eta_{eff}(\omega) d\omega + \\
& \frac{\eta_2^2(1-\eta_1)^2}{4} \int |\Phi|^2(\omega) \eta_{eff}(\omega) d\omega \int |\Phi|^2(\omega') \eta_{eff}(\omega') d\omega', \tag{B.9}
\end{aligned}$$

which will be put back when we calculate the heralding and success probabilities.

Next we interfere telecom photons with B.S. that $\hat{a}_s^{\dagger, A} = \frac{\hat{c}_1^\dagger + \hat{c}_2^\dagger}{\sqrt{2}}$, $\hat{a}_s^{\dagger, B} = \frac{\hat{c}_1^\dagger - \hat{c}_2^\dagger}{\sqrt{2}}$, and again a quantum efficiency $\eta(\omega, \omega_0)$ for telecom photon is introduced. Use $\hat{c}_1^\dagger = \sqrt{1-\eta} \hat{c}_3^\dagger + \sqrt{\eta} \hat{c}_4^\dagger$ and trace over the reflected mode \hat{c}_3^\dagger conditioning on the click of \hat{c}_4^\dagger from NRPD [33]. The effective density matrix becomes

$$\begin{aligned}
\hat{\rho}_{out}^{(2)} &= \int_{-\infty}^{\infty} d\omega_0 \text{Tr}_{c4} \{ \text{Tr}_{c3} [\hat{U}_{BS}^C \hat{\rho}_{in} \hat{U}_{BS}^{\dagger, C}] \hat{M}_4 \} \\
&\equiv \int_{-\infty}^{\infty} d\omega_0 \hat{\rho}_{out}^{(2)}(\omega_0), \tag{B.10}
\end{aligned}$$

$$\hat{\rho}_{out}^{(2)}(\omega_0) \equiv \text{Tr}_{c4} \{ \hat{\rho}_{in}^{(2)}(\omega_0) \}, \tag{B.11}$$

$$\hat{M}_4 \equiv (\hat{I}_{c4}^\dagger - |0\rangle_{c4} \langle 0|) \otimes |0\rangle_{c2} \langle 0|, \tag{B.12}$$

$$\begin{aligned}
\hat{\rho}_{in}^{(2)}(\omega_0) = & \frac{\eta_1^2(1-\eta_2)^2}{16} \int d\omega_i d\omega'_i \eta_{eff}(\omega_i) \eta_{eff}(\omega'_i) \left\{ \right. \\
& \int d\omega_s (1-\eta(\omega_s)) f(s, i) f^*(s, i') \int d\omega'_s f(s', i') \sqrt{\eta(\omega'_s)} \hat{c}_4^\dagger(\omega'_s) |0\rangle \langle 0| \times \\
& \int d\omega''_s \hat{c}_4(\omega''_s) \sqrt{\eta(\omega''_s)} f^*(s'', i) + \int d\omega_s (1-\eta(\omega_s)) f(s, i) f^*(s, i) \times \\
& \int d\omega'_s f(s', i') \sqrt{\eta(\omega'_s)} \hat{c}_4^\dagger(\omega'_s) |0\rangle \langle 0| \int d\omega'''_s \hat{c}_4(\omega'''_s) \sqrt{\eta(\omega'''_s)} f^*(s''', i') + \\
& \int d\omega'_s (1-\eta(\omega'_s)) f(s', i') f^*(s', i') \int d\omega_s f(s, i) \sqrt{\eta(\omega_s)} \hat{c}_4^\dagger(\omega_s) |0\rangle \langle 0| \times \\
& \int d\omega''_s \hat{c}_4(\omega''_s) \sqrt{\eta(\omega''_s)} f^*(s'', i) + \int d\omega'_s (1-\eta(\omega'_s)) f(s', i') f^*(s', i) \times \\
& \int d\omega_s f(s, i) \sqrt{\eta(\omega_s)} \hat{c}_4^\dagger(\omega_s) |0\rangle \langle 0| \int d\omega'''_s \hat{c}_4(\omega'''_s) \sqrt{\eta(\omega'''_s)} f^*(s''', i') + \\
& \int d\omega'_s \sqrt{\eta(\omega'_s)} f(s', i') \int d\omega_s \sqrt{\eta(\omega_s)} f(s, i) \hat{c}_4^\dagger(\omega_s) \hat{c}_4^\dagger(\omega'_s) |0\rangle \langle 0| \times \\
& \left. \int d\omega''_s \sqrt{\eta(\omega''_s)} f^*(s'', i) \int d\omega'''_s \sqrt{\eta(\omega'''_s)} f^*(s''', i') \hat{c}_4(\omega''_s) \hat{c}_4(\omega'''_s) \right\} + \\
& \frac{\eta_1 \eta_2 (1-\eta_1)(1-\eta_2)}{8} \left\{ \int d\omega_i \eta_{eff}(\omega_i) \int f(s, i) d\omega_s \int f^*(s', i) d\omega'_s \times \right. \\
& \int d\omega |\Phi(\omega)|^2 \eta_{eff}(\omega) \sqrt{\eta(\omega_s)} \hat{c}_4^\dagger(\omega_s) \left(\hat{S}_B^\dagger |0\rangle \langle 0| \hat{S}_B + \hat{S}_A^\dagger |0\rangle \langle 0| \hat{S}_A \right) \times \\
& \hat{c}_4(\omega'_s) \sqrt{\eta(\omega'_s)} \int \int f(s, i) d\omega_s \Phi^*(\omega_i) \eta_{eff}(\omega_i) d\omega_i \times \\
& \left. \int \int f^*(s', i') d\omega'_s \Phi(\omega'_i) \eta_{eff}(\omega'_i) d\omega'_i \sqrt{\eta(\omega_s)} \hat{c}_4^\dagger(\omega_s) \times \right. \\
& \left. \left(\hat{S}_B^\dagger |0\rangle \langle 0| \hat{S}_A + \hat{S}_A^\dagger |0\rangle \langle 0| \hat{S}_B \right) \hat{c}_4(\omega'_s) \sqrt{\eta(\omega'_s)} \right\}, \tag{B.13}
\end{aligned}$$

where a brief notation for spectrum $f(s, i) \equiv f(\omega_s, \omega_i)$ and quantum efficiency $\eta(\omega) \equiv \eta(\omega, \omega_0)$. This quantum efficiency refers to the telecom photon. We proceed to trace over the detected modes and the density matrix can be simplified by interchange of variables in integration.

$$\begin{aligned}
\hat{\rho}_{out}^{(2)}(\omega_0) = & \frac{\eta_1^2(1-\eta_2)^2}{8} \int d\omega_i d\omega'_i \eta_{eff}(\omega_i) \eta_{eff}(\omega'_i) \left\{ \right. \\
& \int d\omega_s (1-\eta(\omega_s, \omega_0)) f(\omega_s, \omega_i) f^*(\omega_s, \omega'_i) \int d\omega'_s f(\omega'_s, \omega'_i) f^*(\omega'_s, \omega_i) \eta(\omega'_s, \omega_0) + \\
& \int d\omega_s (1-\eta(\omega_s, \omega_0)) |f(\omega_s, \omega_i)|^2 \int d\omega'_s |f(\omega'_s, \omega'_i)|^2 \eta(\omega'_s, \omega_0) + \\
& \frac{1}{2} \int d\omega'_s \eta(\omega'_s, \omega_0) |f(\omega'_s, \omega'_i)|^2 \int d\omega_s \eta(\omega_s, \omega_0) |f(\omega_s, \omega_i)|^2 + \frac{1}{2} \times \\
& \left. \int d\omega'_s \eta(\omega'_s, \omega_0) f(\omega'_s, \omega'_i) f^*(\omega'_s, \omega_i) \int d\omega_s \eta(\omega_s, \omega_0) f(\omega_s, \omega_i) f^*(\omega_s, \omega'_i) \right\} |0\rangle \langle 0|
\end{aligned}$$

$$\begin{aligned}
& + \frac{\eta_1 \eta_2 (1 - \eta_1)(1 - \eta_2)}{8} \left\{ \int d\omega_i \eta_{eff}(\omega_i) \int \eta(\omega_s, \omega_0) |f(\omega_s, \omega_i)|^2 d\omega_s \times \right. \\
& \int d\omega |\Phi(\omega)|^2 \eta_{eff}(\omega) \left(\hat{S}_B^\dagger |0\rangle \langle 0| \hat{S}_B + \hat{S}_A^\dagger |0\rangle \langle 0| \hat{S}_A \right) + \\
& \int \int \eta(\omega_s, \omega_0) f(\omega_s, \omega_i) d\omega_s \Phi^*(\omega_i) \eta_{eff}(\omega_i) d\omega_i \int f^*(\omega_s, \omega'_i) \Phi(\omega'_i) \eta_{eff}(\omega'_i) d\omega'_i \times \\
& \left. \left(\hat{S}_B^\dagger |0\rangle \langle 0| \hat{S}_A + \hat{S}_A^\dagger |0\rangle \langle 0| \hat{S}_B \right) \right\}, \tag{B.14}
\end{aligned}$$

where the trace over two photon states requires the commutation relation of photon operators.

$$\begin{aligned}
& \text{Tr}[\hat{m}_4^\dagger(\omega_s) \hat{m}_4^\dagger(\omega'_s) |0\rangle \langle 0| \hat{m}_4(\omega''_s) \hat{m}_4(\omega'''_s)] \\
& = \langle 0| \hat{m}_4(\omega''_s) [\delta(\omega_s, \omega'''_s) + \hat{m}_4^\dagger(\omega_s) \hat{m}_4(\omega'''_s)] \hat{m}_4^\dagger(\omega'_s) |0\rangle, \\
& = \delta(\omega_s, \omega''_s) \delta(\omega'_s, \omega'_s) + \delta(\omega_s, \omega'_s) \delta(\omega'_s, \omega'''_s). \tag{B.15}
\end{aligned}$$

The above is the general formulation for the un-normalized density matrix conditioning on three clicks of NRPD's. We've included spectral quantum efficiency of the detector either for near-infrared (η_{eff}) or telecom wavelength ($\eta_t \equiv \int_{-\infty}^{\infty} \eta(\omega, \omega_0) d\omega_0$)

To proceed, we assume a flat and finite spectrum response ($\eta_{eff}(\omega) = \eta_{eff}$, $\eta_t(\omega) = \eta_t$) with the range $\omega_0 \in [\Omega - \Delta, \Omega + \Delta]$ centered at Ω (near-infrared or telecom) and $\omega \in [\omega_0 - \delta, \omega_0 + \delta]$ [39]. The widths 2Δ and 2δ are large enough compared to our source bandwidth so these detection events do not give us any information of spectrum for our source. A perfect efficiency also means no photon loss during detection. Note that the integral involves multiplication of two telecom photon efficiency $\int_{-\infty}^{\infty} \eta(\omega, \omega_0) \eta(\omega', \omega_0) d\omega_0 = \eta_t^2(\omega)$ that is valid if the source bandwidth is smaller than detector's.

After the integration of ω_0 , we have

$$\begin{aligned}
\hat{\rho}_{out}^{(2)} & = \frac{\eta_1^2 (1 - \eta_2)^2}{8} \eta_{eff}^2 \int d\omega_i d\omega'_i \left\{ (1 - \eta_t) \eta_t \int d\omega_s f(\omega_s, \omega_i) f^*(\omega_s, \omega'_i) \times \right. \\
& \int d\omega'_s f(\omega'_s, \omega'_i) f^*(\omega'_s, \omega_i) + (1 - \eta_t) \eta_t \int d\omega_s |f(\omega_s, \omega_i)|^2 \int d\omega'_s |f(\omega'_s, \omega'_i)|^2 + \\
& \frac{\eta_t^2}{2} \int d\omega'_s |f(\omega'_s, \omega'_i)|^2 \int d\omega_s |f(\omega_s, \omega_i)|^2 + \frac{\eta_t^2}{2} \int d\omega'_s f(\omega'_s, \omega'_i) f^*(\omega'_s, \omega_i) \times \\
& \left. \int d\omega_s f(\omega_s, \omega_i) f^*(\omega_s, \omega'_i) \right\} |0\rangle \langle 0| + \frac{\eta_1 \eta_2 (1 - \eta_1)(1 - \eta_2)}{8} \eta_t^2 \eta_{eff}^2 \times \\
& \left\{ \int d\omega_i \int |f(\omega_s, \omega_i)|^2 d\omega_s \int d\omega |\Phi(\omega)|^2 \left(\hat{S}_B^\dagger |0\rangle \langle 0| \hat{S}_B + \hat{S}_A^\dagger |0\rangle \langle 0| \hat{S}_A \right) + \right. \\
& \int \int f(\omega_s, \omega_i) d\omega_s \Phi^*(\omega_i) d\omega_i \int f^*(\omega_s, \omega'_i) \Phi(\omega'_i) d\omega'_i \\
& \left. \left(\hat{S}_B^\dagger |0\rangle \langle 0| \hat{S}_A + \hat{S}_A^\dagger |0\rangle \langle 0| \hat{S}_B \right) \right\}. \tag{B.16}
\end{aligned}$$

Appendix B.3. Density Matrix of PME Projection and Quantum Teleportation

In Section 4.2, we have the normalized density operator $\hat{\rho}_{out,n}^{(2),AB}$ of the DLCZ entangled state through entanglement swapping. With another pair of DLCZ entangled state, $\hat{\rho}_{out,n}^{(2),CD}$, the joint density operator for these two pairs constructs the polarization maximally entangled state (PME) projection and is interpreted as

$$\begin{aligned} \hat{\rho}_{out,n}^{(2),AB} \otimes \hat{\rho}_{out,n}^{(2),CD} = & \frac{1}{(a+b)^2} \left\{ a^2 |0\rangle\langle 0| + \frac{ab}{2} \left[|0\rangle_{AB}\langle 0| \left(\hat{S}_C^\dagger |0\rangle\langle 0| \hat{S}_C + \hat{S}_D^\dagger |0\rangle\langle 0| \hat{S}_D \right. \right. \right. \\ & + \lambda_1 \hat{S}_C^\dagger |0\rangle\langle 0| \hat{S}_D + \lambda_1 \hat{S}_D^\dagger |0\rangle\langle 0| \hat{S}_C \left. \left. + |0\rangle_{CD}\langle 0| \left(\hat{S}_B^\dagger |0\rangle\langle 0| \hat{S}_B + \hat{S}_A^\dagger |0\rangle\langle 0| \hat{S}_A \right. \right. \right. \\ & + \lambda_1 \hat{S}_B^\dagger |0\rangle\langle 0| \hat{S}_A + \lambda_1 \hat{S}_A^\dagger |0\rangle\langle 0| \hat{S}_B \left. \left. \right] + \frac{b^2}{4} \left(\hat{S}_C^\dagger |0\rangle\langle 0| \hat{S}_C + \hat{S}_D^\dagger |0\rangle\langle 0| \hat{S}_D \right. \right. \\ & + \lambda_1 \hat{S}_C^\dagger |0\rangle\langle 0| \hat{S}_D + \lambda_1 \hat{S}_D^\dagger |0\rangle\langle 0| \hat{S}_C \left. \left. \right) \otimes \left(\hat{S}_B^\dagger |0\rangle\langle 0| \hat{S}_B + \hat{S}_A^\dagger |0\rangle\langle 0| \hat{S}_A \right. \right. \\ & \left. \left. + \lambda_1 \hat{S}_B^\dagger |0\rangle\langle 0| \hat{S}_A + \lambda_1 \hat{S}_A^\dagger |0\rangle\langle 0| \hat{S}_B \right) \right\}, \end{aligned} \quad (B.17)$$

which is used to calculate the success probability after post measurement [a click from each side, the side of (A or C) and (B or D)]. $a = \eta_r(2 - \eta) \left(1 + \sum_j \lambda_j^2 \right)$, $b = 4$, and $\eta_r = \eta_1/\eta_2$, $\eta = \eta_t$, λ_j is Schmidt number that is used to decompose the two-photon source from the cascade transition.

In DLCZ protocol, quantum teleportation uses the similar setup in PME projection and combines with the desired teleported state, $|\Phi\rangle = (d_0 \hat{S}_{I_1}^\dagger + d_1 \hat{S}_{I_2}^\dagger)|0\rangle$, which is represented by two other atomic ensembles I_1 and I_2 . The requirement of normalization of the state is $|d_0|^2 + |d_1|^2 = 1$, and the density operator of quantum teleportation is $\hat{\rho}_{QT} = |\Phi\rangle\langle\Phi| \otimes \hat{\rho}_{out,n}^{(2),AB} \otimes \hat{\rho}_{out,n}^{(2),CD}$. Conditioning on clicks of \hat{D}_{I_1} and \hat{D}_{I_2} , the effective density matrix for quantum teleportation is (using $\hat{S}_{I_1}^\dagger = (\hat{D}_{I_1} + \hat{D}_A)/\sqrt{2}$, $\hat{S}_{I_2}^\dagger = (\hat{D}_{I_2} + \hat{D}_C)/\sqrt{2}$ for the effect of beam splitter)

$$\begin{aligned} \hat{\rho}_{QT,eff} = & \left[\frac{|d_0|^2}{2} (\hat{D}_{I_1}^\dagger |0\rangle\langle 0| \hat{D}_{I_1}) + \frac{|d_1|^2}{2} (\hat{D}_{I_2}^\dagger |0\rangle\langle 0| \hat{D}_{I_2}) + \frac{d_0 d_1^*}{2} (\hat{D}_{I_1}^\dagger |0\rangle\langle 0| \hat{D}_{I_2}) \right. \\ & \left. + \frac{d_0^* d_1}{2} (\hat{D}_{I_2}^\dagger |0\rangle\langle 0| \hat{D}_{I_1}) \right] \otimes \frac{1}{(a+b)^2} \left\{ a^2 |0\rangle\langle 0| + \frac{ab}{2} \left[|0\rangle_{AB}\langle 0| \right. \right. \\ & \left(\frac{\hat{D}_{I_2}^\dagger |0\rangle\langle 0| \hat{D}_{I_2}}{2} + \hat{S}_D^\dagger |0\rangle\langle 0| \hat{S}_D + \lambda_1 \frac{\hat{D}_{I_2}^\dagger}{\sqrt{2}} |0\rangle\langle 0| \hat{S}_D + \lambda_1 \hat{S}_D^\dagger |0\rangle\langle 0| \frac{\hat{D}_{I_2}}{\sqrt{2}} \right) \\ & + |0\rangle_{CD}\langle 0| \left(\hat{S}_B^\dagger |0\rangle\langle 0| \hat{S}_B + \frac{\hat{D}_{I_1}^\dagger |0\rangle\langle 0| \hat{D}_{I_1}}{2} + \lambda_1 \hat{S}_B^\dagger |0\rangle\langle 0| \frac{\hat{D}_{I_1}}{\sqrt{2}} + \lambda_1 \frac{\hat{D}_{I_2}^\dagger}{\sqrt{2}} |0\rangle\langle 0| \hat{S}_B \right) \left. \right] \\ & + \frac{b^2}{4} \left(\frac{\hat{D}_{I_2}^\dagger |0\rangle\langle 0| \hat{D}_{I_2}}{2} + \hat{S}_D^\dagger |0\rangle\langle 0| \hat{S}_D + \lambda_1 \frac{\hat{D}_{I_2}^\dagger}{\sqrt{2}} |0\rangle\langle 0| \hat{S}_D + \lambda_1 \hat{S}_D^\dagger |0\rangle\langle 0| \frac{\hat{D}_{I_2}}{\sqrt{2}} \right) \otimes \\ & \left. \left(\hat{S}_B^\dagger |0\rangle\langle 0| \hat{S}_B + \frac{\hat{D}_{I_1}^\dagger |0\rangle\langle 0| \hat{D}_{I_1}}{2} + \lambda_1 \hat{S}_B^\dagger |0\rangle\langle 0| \frac{\hat{D}_{I_1}}{\sqrt{2}} + \lambda_1 \frac{\hat{D}_{I_2}^\dagger}{\sqrt{2}} |0\rangle\langle 0| \hat{S}_B \right) \right\}, \end{aligned} \quad (B.18)$$

which is used to calculate the success probability for teleported state.

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